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OF

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BRITISH MUSEUM

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CONDITIONS OF THE KEITH PRIZE.

This Prize, the interest of a sum which now amounts to about £800, left by the late Alexander Keith, Esq. of Ravelston and Dunnottar, will be awarded by the President and Council of the Royal Society of Edinburgh, on the following conditions:—

I. The author of the best paper on a scientific subject (preference being, in all cases given to a paper containing an important discovery in science made in any part of the world), communicated in the first instance to the Royal Society during the sessions 1851–2, 1852–3, or any two succeeding sessions, shall be entitled to the biennial interest of the Keith Fund, accruing in the respective periods.*

II. The form of the Prize shall be a Gold Medal, of not more than Fifteen Guineas value. The remainder of the sum shall be given in money, to be spent in Plate or otherwise, at the discretion of the receiver.

III. The award being duly intimated to the receiver of the Prize, he is to apply forthwith to the Treasurer of the Society for payment of it; and, failing to do so within six months of the date of the intimation, he shall forfeit the money, but shall be entitled to receive the Medal.

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* The proceeds of all preceding biennial periods have, in accordance with the decision of the President and Council, been either awarded to scientific individuals, or added to the capital sum.

LAWS

OF THE

ROYAL SOCIETY OF EDINBURGH.

SEPTEMBER, 1853.

BRITISH MUSEUM

LAWS.

I.

THE ROYAL SOCIETY OF EDINBURGH shall consist of Ordinary and Honorary Fellows.

II.

Every Ordinary Fellow, within three months after his election, shall pay Five Guineas as fees of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer.*

III.

All Fellows who shall have paid Twenty-five years' annual contribution shall be exempt from farther payment.

IV.

The fees of admission of an Ordinary Non-Resident Fellow shall be £26, 5s., payable on his admission; and in case of any Non-Resident Fellow coming to reside at any time in Scotland, he shall, during each year of his residence, pay the usual annual contribution of £3, 3s., payable by each Resident Fellow; but after payment of such annual contribution for eight years, he shall be exempt from any farther payment.

V.

Members failing to pay their contribution for three successive years (due application having been made to them by the Treasurer) shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

VI.

None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-Bearers, or interfere in the patrimonial interests of the Society.

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* A modification of this rule, in certain cases, was agreed to 3d January 1831.

VII.

The number of Ordinary Fellows shall be unlimited.

VIII.

The Ordinary Fellows, upon producing an order from the TREASURER, shall be entitled to receive from the Publisher, gratis, the Parts of the Society's Transactions which shall be published subsequent to their admission.

IX.

No person shall be proposed as an Ordinary Fellow, without a recommendation subscribed by *One* Ordinary Fellow, to the purport below.* This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall afterwards be printed in the circulars for three Ordinary Meetings of the Society, previous to the day of the election, and shall lie upon the table during that time.

X.

Honorary Fellows shall not be subject to any Contribution. This class shall consist of persons eminently distinguished for science or literature, Its number shall not exceed Fifty-six, of whom twenty may be British subjects, and thirty-six may be subjects of foreign states.

XI.

Personages of Royal Blood may be elected Honorary Fellows, without regard to the limitation of numbers specified in Law X.

XII.

Honorary Fellows may be proposed by the Council, or by a recommendation (in the form given below)† subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from the Chair at one meeting, and printed in the circular for the meeting at which the ballot is to take place.

---

* " A. B., a gentleman well skilled in several branches of Science (*or Polite Literature as the* " *case may be*), being to my knowledge desirous of becoming a Fellow of the Royal Society of Edin-" burgh, I hereby recommend him as deserving of that honour, and as likely to prove a useful and " valuable Member."

This recommendation to be accompanied by a request of admission signed by the Candidate.

† We hereby recommend ________________________________

for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

____

____

____

To the President and Council of the Royal Society
of Edinburgh.

XIII.

The election of Ordinary Fellows shall take place at the Ordinary Meetings of the Society. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote.

XIV.

The Ordinary Meetings shall be held on the first and third Mondays of every month, from November to June inclusive. Regular minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make.

XV.

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the *Transactions* of the Society, and shall superintend the printing of the same.

XVI.

The Transactions shall be published in Parts or *Fasciculi* at the close of each session, and the expense shall be defrayed by the Society.

There shall be elected annually, for conducting the publications and regulating the private business of the Society, a Council, consisting of a President; Six Vice-Presidents, two at least of whom shall be resident; Twelve Counsellors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator, and an Assistant-Curator of the Museum and Library.

XVII.

Four Counsellors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XVIII.

An Extraordinary Meeting for the Election of Office-Bearers shall be held on the fourth Monday of November annually.

XIX.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such meetings.

XX.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the last Ordinary Meeting in January, he shall present the accounts for the preceding year, duly audited. At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXI.

At the Extraordinary Meeting in November, a Committee of three Fellows shall be chosen to audit the Treasurer's accounts, and give the necessary discharge of his intromissions.

The report of the examination and discharge shall be laid before the Society at the last Ordinary Meeting in January, and inserted in the records.

XXII.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes, he shall, when necessary, employ a clerk, to be paid by the Society.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered: they shall specify all the Donations received, and furnish a list of them, and of the donors' names, to the Curator of the Library and Museum: they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty.

XXIII.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the Hall, for the inspection of the Fellows.

XXIV.

All articles of the above description shall be open to the inspection of the Fellows, at the Hall of the Society, at such times, and under such regulations, as the Council from time to time shall appoint.

XXV.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date.

ERRATA.

Page 64, line 23, *for* 48 *read* 56.
... 71, ... 7 from the bottom, *for* 215 *read* 216.
... 74, ... 7 from the top, *for* 20 *read* 21.
... 74, ... 10 from the top, *for* 5·23 *read* 8·23.
... 78, ... 18 from the top, *for* 8·662 *read* 8·62.

ERRATA.

In Mr RANKINE's paper, "On the Centrifugal Theory of Elasticity," Vol. xx., Part iii.

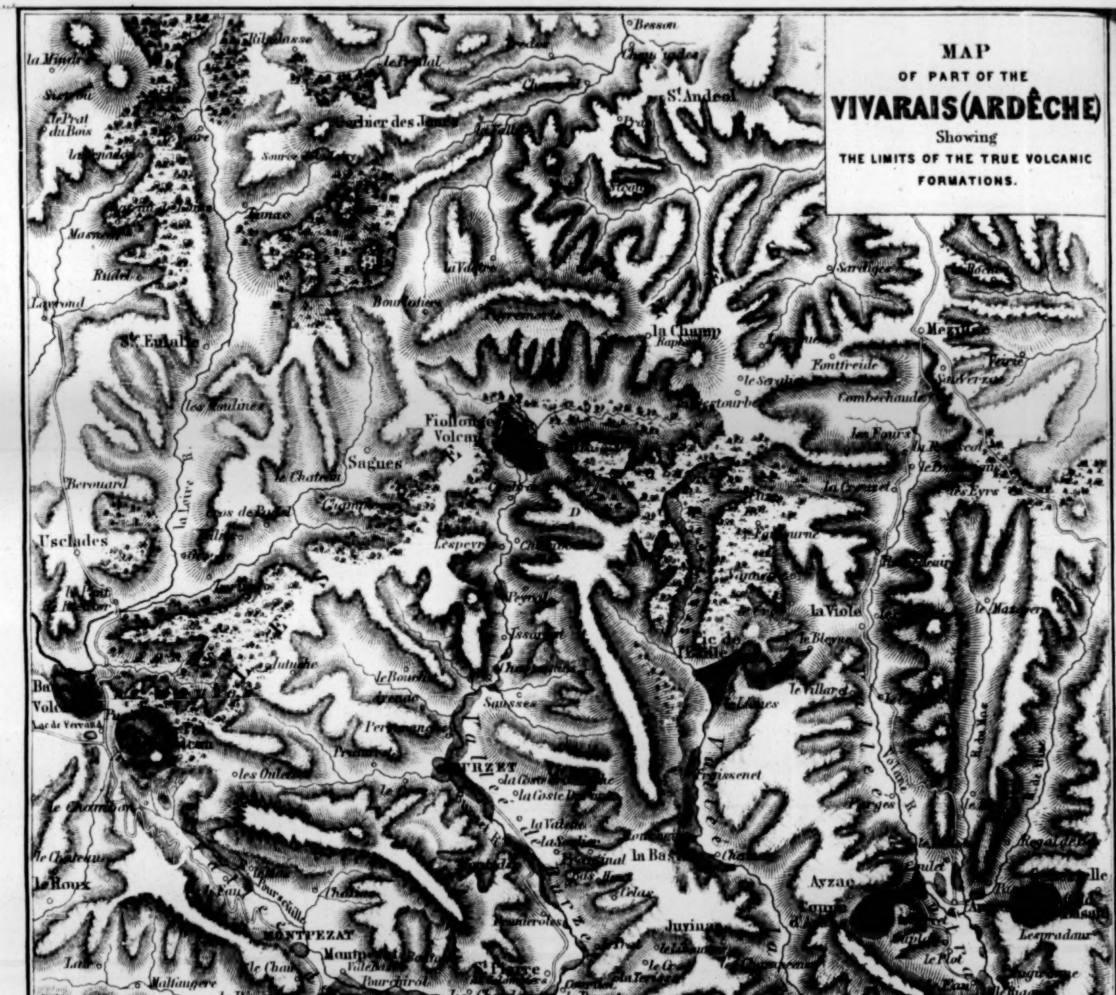
Article 10, page 437, line 7; in Equation 27, *for* $\int (\tau - x) \frac{d p}{d \tau} d V$ *read* $\int \left\{ (\tau - x) \frac{d p}{d \tau} - p \right\} d V$.

Art. 11, page 438, line 12; on each side of the equation, make the same correction.

Art. 12, page 439, second line from the bottom; *for* $J \mu$ *read* $\frac{J}{\mu}$.

In Bishop TERROT's paper, "Summation of a Compound Series, and its Application to a Problem in Probabilities," Vol. xx., Part iv.

Page 541, line 4; *for* $(\overline{m-p} \, . \, \overline{m-p+1} \, . \, . \, . \, . \, \overline{m-p+q+1})$, *read* $(\overline{m-p-q+1} \, . \, \overline{m-p-q+2} \, . \, . \, . \, . \, \overline{m-p})$
Page 545, line 6; *for* 20, *read* 24.



MAP
OF PART OF THE
VIVARAIS (ARDÊCHE)
Showing
THE LIMITS OF THE TRUE VOLCANIC
FORMATIONS.

TRANSACTIONS.

I.—*On the Volcanic Geology of the Vivarais* (*Ardêche*). By JAMES D. FORBES, Esq., F.R.S., Sec. R. S. Ed., Professor of Natural Philosophy in the University of Edinburgh.

(Read 3d and 17th January 1848.)

THE limited district of country which I am about to describe, is one of those which may rank amongst the least frequented in the civilized part of Europe, yet which might justly claim for France the character of romantic beauty which travellers on her beaten highways commonly, and not without reason, deny to her.

The modern department of the Ardêche, corresponding in part to the ancient province of the Vivarais, includes country of very dissimilar features, the southern and eastern part, forming the right bank of the Rhone near Viviers, being comparatively flat; whilst the north-western boundary is the irregular chain of the Cevennes, including the localities more immediately to be described. This chain is not so remarkable for its absolute height, although that be considerable, rising at the Mont Mezenc, in the neighbouring department of the Haute Loire, to an elevation of 5750 English feet above the sea, as from forming the separation of a remarkably elevated tract stretching to the north and west, and which suddenly subsides, at the point of which we now speak, into the wide champaign country of the Lower Rhone, possessing a very different aspect, soil, climate, and population.

The high ground, or *plateau*, of which we have spoken, being thinly peopled, bleak, and steril (in its general character), compared to the fertile and vine-clad banks of the Rhone and Saone, immediately to the eastward, is but little traversed. In fact, only one great road passes through it, the post-road from Paris to Perpignan. It will readily be understood, also, why the Cevennes Mountains themselves are rarely visited, being left between this great road and the more

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BRITISH MUSEUM

usual thoroughfare of the Rhone, down which travellers and tourists glide by thousands, without spending a thought upon the intricate country which lies on their right, their attention being rather attracted to the more striking features of the outlying portions of the Alpine chain on the left, where the noble outlines of the mountains of Dauphiné hold out a more tempting prospect of romantic scenery and of geological interest.

Nevertheless, from a very early period of the revival of geological studies in these latter times, the Provinces of the Ardêche and Haute Loire (the ancient Vivarais and Velay) attracted considerable notice. When men were no longer content to ascribe, with Voltaire, the presence of extraneous fossils in mountain rocks to the passage of pilgrims with their scallop shells,—or with others, the scoriæ of ancient volcanoes, covering an area of many square miles, to the iron forges of the Romans,—they began, very properly, to compare the marks of the most apparently recent changes of the earth's surface, not however belonging to the historic period, with those going on under our own eyes, such as the eruptions of volcanoes, and by clearly establishing identity of effects in some cases, were led to almost irresistible conclusions from analogy in others. Thus the presence of volcanic craters, scoriæ, and lava currents, in several parts of central France, could not be doubted by any one who had seen the burnt ground of Etna or Vesuvius. The phenomena should rather be called identical than analogous; but the argument of analogy from the lavas thus detected, to the volcanic origin of lava-like stone or basalt in many countries where no red scoriæ, no declivitous streams of once melted matter, and no indubitable craters could be found, opened up a field for more prolonged study, and more cautious generalization. In this department Faujas de St Fond, a native of the Vivarais, distinguished himself; and with great industry, and considerable, though not invariable judgment, he set about identifying the features of the unquestionable lava streams of his own country with more obscure, because more ancient ejections of melted matter forming rocks on the surface, not only there, but in distant countries, particularly in Great Britain, whither he made a journey on purpose. To establish indubitably the connection of basalts with lavas was the main object of his large work in folio, on the Extinct Volcanoes of the Vivarais and Velay, published in 1778,* which contains observations of merit, and descriptions generally exact, notwithstanding the rudeness of the engravings by which they are illustrated. This work contains, perhaps, the most complete description of the volcanic district of the department of the Ardêche which has yet been published, and the ample narrative of Faujas has been the guide of every subsequent explorer (and they have not been numerous) of this remarkable country. The circumstance of proximity of situation, which led Faujas to explore the hills of his own neighbourhood (for he resided at Montelimart, exactly upon the opposite bank of the Rhone), subsequently turned the current of attention to a different district, more accessible to tourists, nearer to Paris, and in the close vicinity of a great provincial town. Clermont and its environs, including the Puy de Dôme, naturally withdrew geologists from the remoter and more scattered volcanic features of the Southern Cevennes, and so much has been written and published upon Auvergne proper, as to render any attempt at addition (at least in the way of general description) altogether superfluous. M. BERTRAND's accurate local descriptions and map of the singular basin of Le Puy (the ancient Velay), and the masterly pencil of Mr SCROPE, have, in a great measure, exhausted the descriptive geology of that most curious, but most difficult field of study. An easier, but rather more neglected subject remained in the province of the Vivarais, the favourite ground of FAUJAS, to which I acknowledge that I was first attracted by the panorama of the basaltic colonnade of Jaujac in Mr SCROPE's incomparable atlas. Having previously inspected, for my own instruction, the other four great volcanic centres of this region of France, viz., the Monts Dôme, the Monts D'Or, the Cantal, and Le Puy, I meant to finish, as my predecessors had done, with a hasty survey of Vivarais. But I found there a united attraction of scenery and geology, together with that isolation and remoteness which lends a peculiar, though doubtless a selfish charm to a prize which we imagine that others have, in some degree, overlooked, which caused me to fix my quarters in the very first village which I reached, and again, two years later (in 1841), to revisit every point of geological interest, to extend my notes, and to prepare a map and drawings of the volcanic phenomena. These were intended to have been at once reduced into a digested form, and published in the Transactions of one of our Societies; but, in the same year, a fresh subject of interest was opened to me, and for a time withdrew my attention entirely from any other voluntary pursuit requiring much leisure; and since that time the theory of glaciers has occupied nearly all my spare moments.* I now resume my original intention of describing the ancient volcanoes of the Vivarais, with the hope of being able to infuse into the general reader some small share of the admiration with which my first visit filled me, and which a second did not abate. I shall first describe the track by which I originally entered these valleys, as serving to point out the circumstances of contrast to which I have above alluded.

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* Recherches sur les Volcans eteints du Vivarais et du Velay, avec un discours sur les volcans brûlans, &c. Grenoble et Paris, 1778.

The spring of 1839 was late and cold, and in France and the basin of Le Puy, a town situated 625 metres, or nearly 2000 English feet above the sea,† was not the first place to feel the influence of summer. Notwithstanding its great absolute height, the country rises still higher in every direction, save the narrow gorge by which the river Loire struggles out of the circuit of lofty hills forming three or four pretty distinct ranges which enclose the basin, and which contribute to injure its climate; for there is little wood, and the winds from every quarter sweep unchecked over the extensive and bare *plateaux* of granite and basalt of which it is chiefly composed.* On *the* 25*th May* I was exposed to a piercing wind, having a temperature of 36°, with frequent showers of hail. A few days later I engaged a horse to carry me across the highest part of the range of the Cevennes, and the sources of the Loire, down to the valleys of the Ardêche. My kind friend M. Bertrand had given me a route which should embrace the most interesting geological points; and with his map as a guide, I started alone with a pony, to sleep at the foot of the Mont Mezenc, at a village bearing the unpromising name of Fay-le-froid. Though it was the last day but one of May, little appearance of spring was visible; indeed, it had hardly an opportunity of making any impression on the singularly bare and rugged features of a country nearly destitute of trees, and often covered for miles with brown angular fragments of basalt.

---

* These pages were written in 1847.

† BERTRAND.

The position of Fay-le-froid, a meagre village of little more than a single street, is somewhat greener; it lies upon the northern slope of the Mont Mezenc, and near the church is a bed of basalt, containing fragments of trachyte and granite, with abundance of olivine. Early next morning I was on the gentle ascent which leads in two hours to the summit of the Mont Mezenc. A fog prevented me from enjoying the view,† but it afterwards cleared away sufficiently to enable me to examine the geological section below the point called *La Croix de Boutières*, which is 800 feet vertically lower than the top of the Mezenc, and is situated a little to the south of it. Here the escarpment to the south-east forms a sort of imperfect amphitheatre, in which the Salliouse (a rivulet joining the Ericux, a tributary of the Rhone) takes its rise. This hollow, sometimes called *Le Cirque de Clusels*, presents a section which has justly obtained some celebrity amongst geologists. The peculiarity which it presents is the undoubted *superposition* of trachyte and phonolite or clinkstone (which are felspathic lavas), to common basalt and vesicular scoriæ. The section which I obtained, and from which I took specimens, is shewn in Plate II., fig. 1. There is no proper superposition of trachyte to phonolite; the latter appears to pass into the former, and sometimes to form veins in it. I have no doubt that the trachyte is scorified and rendered vesicular and ochrey by the heat of the basalt injected from below. This appears to be the utmost which can be legitimately inferred from this section; and it is so far a satisfactory conclusion, since it does not contradict the generally established view of the posteriority of basaltic to felspathic lavas, which rule besides receives, in this immediate neighbourhood, so unquestionable a support from

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* See Mr Scrope's Panorama from the Montagne d'Ours.

† In clear weather Mont Blanc is visible from hence. *Bertrand*, *Description du Puy*, *&c.*, p. 124.

Fig 1 p 4 The Mezenc

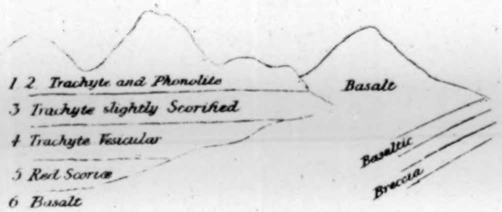
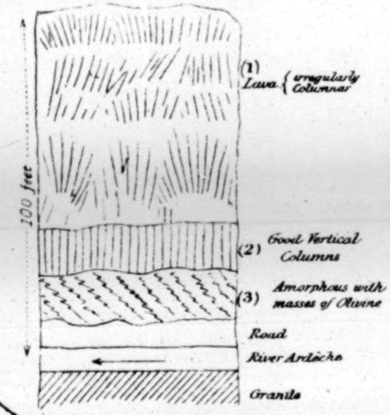
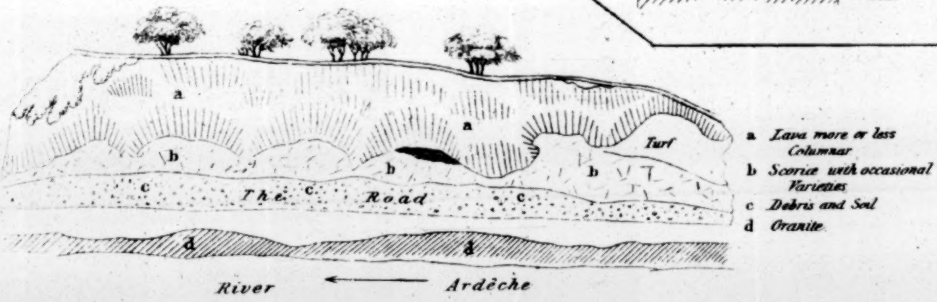


Fig 2 p 11 Near Pont de la Beaüme



100 feet
(1) Lava { irregularly Columnar
(2) Good Vertical Columns
(3) Amorphous with masses of Olivine
Road
River Ardèche
Granite

Fig 3 p 9 Pont de la Beaüme



a
a
b
b
b
c
c
c
The Road
d
d
Turf
a Lava more or less Columnar
b Scoriæ with occasional Varieties
c Debris and Soil
d Granite
River Ardèche

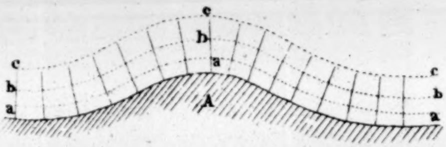


Fig. 4 p. 10

Fig 6 p 15

Section of Lava-Cliff of the Alignon

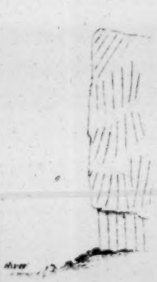


Fig. 7 p. 15 & 29

Near Burzet

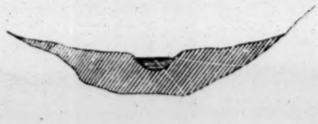
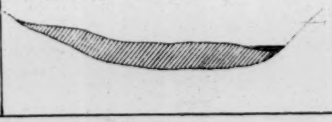


Fig. 8 p. 15



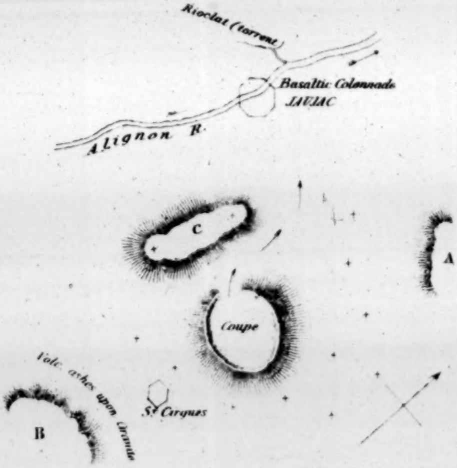


Fig 5 p 16 Environs of Jaujac

Explanation .... A Gneiss with Granite veins. B Primitive, with Volcanic facing C Hill of Sandstone + + + points marking the continuity of the Sandstone → Course of the Lava

W. & A.K. Johnston lith. Edin.

BRITISH MUSEUM

the fragments of trachyte found, as already mentioned, completely imbedded in the basalt, near the church at Fay-le-froid.

That the Mountain of Mezenc is situated near what was once a centre of volcanic action is highly probable, not only from being the most elevated summit of the whole country, and itself composed of volcanic rocks, but from the manner in which the basaltic *plateaux* which have since been broken up, and in great part removed by denuding causes, may be traced from the very environs of Le Puy up to this chain, and also from the peculiar evidences of local fire which the torrefied materials exhibit in the section just presented. I am, however, disposed to agree with those geologists who consider that since the production (perhaps in a good measure by *elevation*) of the Mont Mezenc, the *contour* of the ground has been so completely altered and disfigured as to leave no ground for inferring that we are to trace in its lineaments the actual point of ejection, still less that the *Cirque de Clusels* or *Boutières* is really a " crater of elevation," for it appears to want the essential characters of a crater at all. It would be difficult to prove that the strata (ill-defined even where they exist) dip away universally from a common centre; for only the northern and western portions of the supposed circus can be found; and the argument which has been drawn* from the precipitousness of the rocks in this place compared to their gentle slope in other directions is worth little, since the same reasoning would apply to the whole eastern flank of the range, between Mont Mezenc and the Gerbier des Joncs, compared to the western, which depends upon some general, but probably posterior cause of denudation of which I can give no account. That the expansive action of the basalt, whilst fluid, elevated the previously existing beds of trachyte, and thus contributed to give to the Mezenc its present height, I think there can be no reasonable doubt; and the dikes of basalt occurring here and elsewhere appear to confirm this opinion. It is, however, worthy of note, that if this be cited as an argument in favour of " elevation craters," and as confirming the usual chronology of volcanic rocks of different mineral characters, in the Cantal at least, we find a mountain of phonolite, the Puy Griou, subjacent to the basalts which it is supposed to have elevated.

Whilst I admit, with M. BERTRAND, that immense lava streams proceeded from *some point* or fissure near the Mont Mezenc, and flooded the fresh-water formations of a great part of the basin of Le Puy, I should hesitate before ascribing, with Mr SCROPE, the immense basaltic plateaux of the Coyrons in the lower Vivarais, with the least degree of certainty, or even of probability, to an eruption of the Mezenc.† Amongst other arguments, the numerous *dikes* of basalt, traversing granite, and other formations in this country, seem to shew that the outlets of melted matter were far more widely spread than this idea of a central volcano would permit. Perhaps it may added that the close proximity of the granite to the surface, wherever the volcanic materials have been wasted sufficiently to expose it, and *that* particularly in the valley of the Salliouse, as shewn in M. BERTRAND's Geological Map, almost close to the *Cirque de Clusels*, seems at variance with the supposition that Mont Mezenc is to be regarded as the sole volcanic centre which gave rise to such widely-spread phenomena.

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* *Burat* Terrains Volcaniques de la France Centrale, p. 230.

† Mr SCROPE's General Map gives an altogether erroneous idea of the proximity and mutual dependence of the basaltic formations of the Mezenc and those of the Coyrons.

I must still more emphatically dissent from the theory that the hills of phonolite, forming the north-east barrier of the basin of Le Puy, are the relics of a felspathic eruption proceeding from the Mezenc.* Not to multiply arguments against so very bold, and, in itself, I must think, so improbable a supposition, I will only observe, *first*, the rarity of any appearances, in trachytic mountains, indicating that the matter of which they are composed has been sufficiently liquid to *flow*, after the manner of a current, over any extent of country. *Secondly*, the absence of any appearance of a current in the chain of detached sugar-loaf shaped eminences here referred to. *Thirdly*, that if it were a current, it would not have occupied the *axis* of a granitic elevation, constituting one of the orographic features of the country; and, *lastly*, that the direction of the chain, coincident with that of other important chains, and especially of the chain separating the Allier and the Loire, and the chain of La Margeride, beyond the former, evidently points out an axis of elevation which, in other instances, in this singular country, is the fertile source of local explosions and eruptions, to which these insulated phonolitic peaks may, in my opinion, be, with far more probability, ascribed.

The route, southwards from the Mezenc, presents some singular features. We follow a ridge, sometimes composed of trachyte or phonolite, sometimes of basalt, which separates the gentle slopes towards the basin of the Loire and the more precipitous ones towards that of the Rhone. The views, in the latter direction, are eminently singular, and even romantic; a country intersected with deep ravines, and divided by deep hilly ranges, often capped by fantastic summits, stretches away for many miles in the direction of Chalançon and La Voulte. Some idea of the scenery may be formed from the sketch in Plate V., fig. 5, taken from a spot commanding a view *towards* the Mezenc, *across* the country now referred to. The form of the phonolitic summits (those whose names are given on the drawing) marks their composition in a manner scarcely to be mistaken; but the basis of the whole is granite or gneiss, as pointed out in M. BERTRAND's map, and in the new Geological Map of France. It is probable that this district has been but imperfectly explored; the whole circumference of the Mezenc presents a degree of sterility and desertion almost repulsive. About an hour's walk from the Mezenc is La Clêde, a small public-house where refreshment may be had, and the neighbouring old monastery of Bonnefoi, which belonged to the Chartreux, but is now the remote habitation of a private gentleman, would afford an invaluable centre for excursions to a geologist proposing to examine the neighbourhood. Two hours after leaving the Croix de Boutières, I arrived at the foot of the phonolitic peak of the *Gerbier des Joncs*, which resembles the Pierre de Bar, near Le Puy. The Gerbier des Joncs is best known as the point where the river Loire is understood to take its rise, and where it at once commences its long and tortuous course, a course so involved, that between this point and the defile of Chamalières, by which it issues from the basin of Le Puy, it traverses a length of 250 kilomètres, or 170 English miles, whilst the *direct* distance is not above *one-fifth* part so great.* The height of the source of the Loire is 4505 English feet, which is about 2900 above the defile just mentioned.

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* "The uniformly progressive declination of this series of phonolitic summits from the Mezenc to the bed of the river where they terminate, proves them, in my opinion, to be the remains of a single enormous lava current prior in date to the excavation of the actual channel of the Loire, and far the most considerable in bulk and extent of any which I have had occasion to observe in the phlegræan fields of France."—SCROPE's *Geology of Central France*, p. 129.

After passing the villages of St Eulalie and Usclades, I ascended through the forest of Bauzon, at the foot of the volcanic cone of the same name,† which I afterwards examined more particularly. A moderate descent brought me to the head of the valley of the Fontaulier, and to the singular volcanic crater of Pal, which forms a cup in the midst of granite mountains, never having raised a cone of ashes. The road then rises slightly to the col or passage separating the sources of the Fontaulier from those of the Pourseuille, which descends to the valley of Montpezat. From this point a magnificent view opens. A steep descent of 2000 feet leads to the village of Montpezat, surrounded with verdure, and placed at the entrance of the Bas Vivarais, a perfect contrast to the cheerless highlands of the Velay. It was almost like a peep from the Alps into the warm valleys of Italy, or like some of the pleasant scenes in the Pyrenees. In fine contrast with the deep green of the chestnut-clad slopes, rose the warm reddish-brown summit of the Gravenne of Montpezat, a volcano so fresh in its appearance as to seem as if scarcely yet extinct. Immediately on the left, also, appeared volcanic relics still fresh and cindery, which contrasted with the sombre hue of the granite rocks on which they were spread. From Montpezat almost every part of the volcanic district of the Vivarais can be conveniently reached; and, in point of accommodation, there is not much choice elsewhere, and probably it may be long before it is improved.

But it will assist the clearness of my descriptions if I commence my account of the volcanoes and the surrounding scenery, not from the *centre* of the district but from one extremity; and, as these all lie on or near some of the numerous streams which rise amidst the heights of the Cevennes, and which unite a good many miles to the south of Montpezat to form one considerable river,—the Ardêche, which gives its name to the department,—it will be most convenient to suppose the traveller ascending the Ardêche from the town of Aubenas, one of the most considerable in the province, and arriving at a point called Pont de la Beaume (see the map), where the river Fontaulier, whose rise we have taken notice of, joins the Ardêche, having previously passed close to the village of Montpezat, and immediately under the volcano of the same name, subsequently receiving two minor tributaries, which we shall presently have to describe.

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* BURAT, p. 158.

† From this point of the description, the map, Plate I., may be consulted.

From the neighbourhood of the Pont de la Beaume, the greater part of the valleys of which we are to speak diverge almost like the rays of a fan. As we look *up* the course of the Ardêche, Montpezat occupies nearly the centre of the fan. Now, in all these valleys there is a remarkable uniformity of constitution, and, in some respects, of general appearance. The substratum of the whole is a primitive rock, granite, or in some places gneiss. The distinction is not very important in connection with the phenomena which we are to describe; and I have not attempted to determine the limits of the more crystalline granites, as distinguished from those whose slaty structure may allow them to be considered as having a regular cleavage and direction of beds, subordinate to which hornblende slate also occurs. When I speak, therefore, of granite forming the predominant rock (*uncoloured* on the map), I would not be understood to do so always with precise mineralogical accuracy. So far as I know, there is no peculiarity in the volcanic action in the granitic districts, compared with that in gneiss. A small patch of the coal-formation appears near Jaujac on the Alignon, but it is surrounded by granite or gneiss, which again is succeeded by the lias or oolite formations near Aubenas. The coal-formation occurs in patches nearly all round the great primitive *plateau* of central France. It is extensively worked on the east side at St Ettienne; and, in some places at least, its strata lie horizontally against the granite, shewing the anterior date of the elevation of the latter.* The valleys we have to describe farther agree in this extraordinary particular,—that, *as surely as they contain water they contain a stream of lava or basalt, or the remains of one*, which stream has accommodated itself perfectly to the sinuosities of the channel of primitive rock in which it has run, the possession of which it contests yard by yard with the water; these lava streams are sometimes attenuated to a surprising degree, leaving but small relics for the space of miles; in other places they accumulate to an astonishing thickness and breadth, altering the configuration of the valley, the stagnant pool of lava having, in the first instance, created a lake of water, and compelling the river to alter its course and to excavate a new channel. The tributary of each valley commonly unites with others, accumulating near the points of junction; but the heat being gradually spent, the currents have lost their mobility, and do not at all extend themselves into the plains. When we trace these lava currents to their sources, the result is uniformly the discovery of a crater, often formed in the breast of a mound of cinders, whose fiery-red colour will bear a comparison, in point of apparent freshness, with any of those which stud the flanks of Vesuvius, or the more prolific Etna; and, in very many instances, the precise point of ejection of the lava may be ascertained with the most extreme nicety, and all the accidents of its subsequent course chronologically traced. Thus, every individual eruption has written, as it were, its own history, although the relative dates cannot always be determined. It is an inquiry not a little interesting (at least upon the spot) to collect these rude hieroglyphics, which form a chapter of the ancient records of our globe, and register events amongst the most recent of *geological* change, yet of which no trace or tradition is to be found amongst the histories of the Gallic nations.

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* Burat, p. 4.

The hamlet of La Beaume lies on the right bank of the Ardêche, almost under the shadow of a basaltic colonnade, which stretches parallel to the course of the river, but leaves a level space between the foot of the cliff and the water, along which the public road passes. The section (Plate II., fig. 3) is sufficiently remarkable to have attracted attention ever since the days of FAUJAS ST FOND, who has given a view of it, but with great inaccuracy and exaggeration. Our figure shews the lava stream (*a*) invested with a coating of soil, on which is abundant vegetation. The lava rests on a mass of scoriæ *b*, which again reposes upon the debris and vegetable mould *c*, thus marking strongly the comparative recency of the eruption which produced the lava, and the perfectly *natural* and *modern* condition of the valley into which it flowed. The debris and soil rest upon the primitive rock (granite or gneiss) which is exposed in the bed of the river Ardêche.

The contact of the lava and scoriæ (beds *a* and *b*) presents some interesting considerations; and, *first*, to account for the so frequent phenomenon of the superposition of lava to the scoriæ, which, being its scum and refuse, we should rather expect to find upon its surface, we must recall the peculiar manner of progression of those highly viscid lavas, which most abound in scoriæ (for, whilst very fluid, there is little or no scoria deposited). The progression is exceedingly slow, and, according to the usual laws of a tenacious fluid, moving over a rough surface with great friction, the surface moves faster than the bottom of the stream, and the *front* of the wave of lava (that which would be presented to a spectator towards whom it is in the act of descending) being hard and scoriaceous, in consequence of long exposure to the cooling action of the air, is continually thrust *under* the liquid as it slowly struggles on, and its place is supplied by freshly floating scum from the surface, which finally descends the front of the lava wave, and thus, as it were, unrolls a continuous carpet, over which the more liquid stream flows; here and there breaking through the tenacious but partially yielding and crackling crust by which it is imperfectly confined.* This perfectly explains why we should have such a substratum of scoriæ in this place, which is often wanting below the basalt in the higher and steeper valleys where the lava flowed more rapidly, and it also accounts for the arched forms of the contact-surface, which fig. 3 remarkably displays; these vaults being due to the actual rolling of the lava over the more or less ponderous masses of scoriæ deposited from its own surface. The brick-red colour common in scoriæ beneath lava is to be attributed to the intense heat communicated to them by the lava, *after* they have been covered by it. This heat must have been retained for a prodigious space of time. Brick-red tints are usually produced upon minerals subjected to close or confined heat. The casual removal of the scoriæ has, in some places, left grottoes beneath the vaults. *Secondly,* These vaults also present this remarkable peculiarity, that the columnar structure of the lava (which here, as elsewhere, is best developed near its lower surface) conforms so nicely to the contact-surface with the scoriæ, to which it is always nearly perpendicular, as to give quite the appearance of vaulting stones to the covering of the grottoes. This is a fact so general amongst the lavas of the Vivarais, as to deserve almost to be called universal. It is interesting, as illustrating the development of the *prismatic structure* which was so long supposed to distinguish ancient basalts from true lavas, but which is now universally admitted to characterize both, when the circumstances of cooling are favourable to their production. The fact of the perpendicularity of the columns to the surface of cooling admits of this general illustration,—that if A (Plate II., fig. 4) represents a cold mass of rock overflowed by lava, which gradually loses its heat by contact and conduction, all the points equidistant from the rock, as *a*, *a*, *a*, or *b*, *b*, *b*, or *c*, *c*, *c*, may be conceived to be, at the same moment, in the same condition as regards a tendency to consolidate or to crystallize. Any peculiar action, which depends upon a particular stage of cooling, will therefore affect similarly all the points *a*, *a*, *a*, and so of the rest; that peculiar state of tension which produces the columnar division, will therefore prevail uniformly over any one of these isothermal surfaces (or surfaces equally cooled) at a given time, and will tend to produce its effect everywhere on that surface, and the lines or planes of separation will therefore seem to proceed uniformly from the surface of cooling in a direction perpendicular to it. Or on the less probable hypothesis of the columns being due to real crystallization, the crystals will naturally begin to form at the surface of earliest consolidation, and will shoot outwards, so as to form simultaneously at points equidistant from that surface. The accuracy of the empirical law is greater than any theoretical view could possibly lead us to expect. Varied examples will be quoted in the following pages.

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* That such is the mode of progression of lava streams at a great distance from their origin, or after they have been running for a long time, appears from the descriptions of the best writers on volcanoes. Compare SCROPE's description of the lava of Etna of 1819, in his work on Volcanoes, p. 102; and AULDJO's figure of the descending lava waves of 1831, in his Description of Vesuvius, p. 92.

A little way above La Beaume, near the junction of the rivers of Ardêche and Fontaulier, is the picturesque Castle of Mayras, which, however, offers in itself nothing of much interest. Opposite to this castle (on the other side of the Ardêche) the lava cliff continues from the point already described, so that the colonnade of La Beaume owes its origin to the western, and not to the eastern valleys which unite there. There is little doubt, however, that the lava stream of the eastern valley (or that of Montpezat) may be traced at the sharp turn of the road to Thuez, opposite the Chateau de Mayras, where there is evidently an older and inferior stratum of lava below the greater colonnade. A section at this point, of a cliff about 100 feet in height, is given in fig. 2. The bed No. 3, I take to be the lava of the valley of Montpezat. It contains a remarkable quantity of olivine, and is here amorphous: it is completely detached from the superior mass, which is beautifully columnar where it rests upon the older bed, the columns being vertical, since they rest upon a horizontal base. The beds Nos. 1 and 2 might appear also to belong to distinct currents, the upper part shewing but imperfect columns; a close examination shews, however, that the beds (1) and (2) inosculate in such a manner as to leave no doubt of their being due to one and the same eruption, and that the distinction is caused by the accidental manner of their consolidation.

Valley of the Alignon—Jaujac.

A short distance higher up, the Ardêche divides from its tributary the Alignon, the former descending from Thuez, the latter from Jaujac. The great mass of the lava which we are tracing evidently descends from the latter valley, and here we gain the first view of one of the volcanic vents which has furnished the lavas of the lower valleys, but presents from this point no trace of a crater. It is situated on the ridge separating the valleys of the Ardêche and Alignon, and is sometimes called the Volcano of Neyrac, from a village of the former, or Souillols, from one of the latter. And here occurs an interesting question, whether the great *single* lava-flow, extending to the Pont de La Beaume, is due to this volcano, or to that of Jaujac, higher up the Alignon; for it undoubtedly did not come from the valley of Thuez (on the Ardêche). On my first visit in 1839, I was of opinion that the stream might be traced uninterruptedly from the volcano of Jaujac, and that there was no sufficient evidence that the volcano of Neyrac had yielded a considerable stream into the valley of the Alignon, notwithstanding the presumption afforded by its crater (a considerably decayed one) being broken down (which is commonly due to the pressure of lava) on the south-west side. A very careful examination in 1841 changed my opinion; and, notwithstanding the seeming improbability of the conclusion, we must, I believe, admit that the lava of Jaujac terminates almost exactly where the lava of Neyrac commences, filling the valley to almost the same depth, and with similar matter. And this must be conceded for the following amongst other reasons:—*First*, There is a break in the continuity of the lava stream a little above, and opposite to the village of Souillols, and in the cliff formed by the river. The granite of the country may be traced in the interval. *Secondly*, Up to this point the character of the lava, as displayed continually in the cliff on the right bank of the stream, is remarkably uniform upwards from the point of section (fig. 2) opposite the Castle of Mayras. Only a small portion of the lower part is columnar, surrounded in the greater part of its thickness by basalt nearly amorphous or slightly columnar. The columnar part diminishes in thickness as we ascend the course of the river, and opposite Souillols it is only three or four feet in height. But when the lava cliff reappears after the break alluded to (a break, however, so slight, that it might easily escape notice), it presents a very different front to the river. The cliff is now 130 feet in height,* of which not less than two-thirds, and in some places nearly the whole is composed of a single range of perfectly continuous basaltic pillars; and the level of the prismatic boundary is again gradually depressed, as we approach the undoubted origin of *this* part of the stream, namely, the crater of Jaujac. *Thirdly*, The volcano of Neyrac *does* exhibit a streak of ashes and slag down its southern face. Now the ashes may be seen to pass into the slag, and the slag into the lava of the Alignon near Souillols,—a convincing argument.

We now come to describe more particularly the lava of Jaujac, which extends along the bottom of the valley, from a short distance above the village of that name, to nearly opposite Souillols, at least two miles farther down. All this space has been raised from the natural bed of the stream to a vertical height of perhaps 100 feet on an average, throughout the entire breadth of the valley, and now presents a cultivated and wooded plateau, whose extraneous origin would hardly be suspected but for the deep incision made by the river near the foot of the hills which bound it to the north. This section displays the wonderful colonnade already referred to, of which so correct a representation has been given in Mr Scrope's beautiful work on Central France.

That representation exhibits well the remarkable fact of the gradual *rise* of the lower perfectly-columnar stratum into the higher or imperfect one, in proportion as we follow the section down the valley, or farther from the point of emission (the *Coupe de Jaujac*, near the village of that name). Near Jaujac, where the cliff may be, perhaps, from 60 to 70 feet high, only one-fourth of the height is occupied by the columnar range; down the course of the valley it rises to one-third and one-half of the height of the cliff (which is also increasing), it then divides into two columnar beds, with an amorphous portion between, and finally presents the magnificent display of unbroken columns in almost its entire height, already referred to.

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* Measured by the fall of a stone, and confirmed by the authority of a person who told me that he had measured it with a string.

We have seen, that, sharp as the separation appears of the columns from the amorphous or slightly-columnar part, it is impossible to deny that both are the production of a single eruption. The cause of the abruptness of the separation appears, in the present state of our knowledge, altogether inexplicable. Mr Scrope, indeed, gives the following elucidation, but I doubt whether it can be considered as, in any degree, satisfactory. "[This] singular difference of structure may be accounted for, I conceive, by supposing the elastic vapour, contained in the lava, to have escaped from the upper part, through fissures of retreat, formed in irregular directions as the mass cooled; while the immense pressure, acting on the lower part, would, at the same time, prevent the escape of the contained vapour, and effect its *condensation*, thereby allowing the mass to consolidate in a regular and tranquil manner, such as would facilitate the establishment of straight and vertical axes of contraction, and the formation of very regular hexahedral columnar concretions."* Whatever be the cause, it would appear to become more energetic after the lava has flowed to a considerable distance; and from the regularity with which the limit of the columns follows the sinuosities of the base (see Plate VI., fig. 3), it would seem to depend more upon the rapidity of the cooling action below than upon the superincumbent load of lava.

It is a very curious circumstance that this sharp limit of columnar formations occurs also in circumstances which would appear very different from the volcanic flood of Jaujac, which undoubtedly occurred in the open air exactly like any modern eruption. A recent visit to the Island of Staffa has satisfied me that the basaltic bed in which Fingal's Cave occurs, is constituted precisely like the lavas of the Vivarais, and that the sharp cessation of the columns, and the abrupt transition to a nearly shapeless bed of basalt, is not due to a superposition of other matter, but simply to the limited sphere of action of the crystallizing or concretionary forces; we have there, as at Jaujac, perfect evidence of the contemporaneous formation of both beds; and, indeed, the apparent section of the two would be perfectly identical, the compact part commonly projecting over the columnar part, as in Plate II., fig. 6. Dr MacCulloch, who makes no allusion to the contemporaneity of the beds, or to the cause of their distinction of character, minutely specifies the exact perpendicularity of the principal range of the columns of Staffa to the plane of the stratum which they occupy, which is inclined about 9° to the horizon.* Some of the pillars are a little twisted in their upper parts, although quite straight below, a phenomenon which gives them an appearance of having yielded under the pressure of the load above them; this occurs a little to the east of Fingal's Cave, and is correctly represented in one of Dr MacCulloch's views of the island. If we compare the lava-cliff of Jaujac with Staffa as a mere columnar display, we must give the former a decided preference. The columns are more numerous, more extended, higher, slenderer, better pointed, and, in every respect, more perfect. It is probable that this, as well as their incomparable superiority to any known product of altogether recent volcanoes, is due to the more perfect composition and fusion of the material of which they are formed.

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* Geology of Central France, p. 152.

But this grand phenomenon might have been lost for ever to human sight had not the excavating action of the stream made the *section* which we have been contemplating; and, is it not interesting to inquire in what manner the degradation or destruction of so large a mass of intensely hard rock was effected, and whether it is going on still? Now, as to the present action of the elements, there are abundant proofs that it is going on with great energy. Just below Jaujac, a small stream, named Rioclat, joins the Alignon, and, opposite to their junction, a great mass of the principal lava-cliff fell about three weeks before my visit in 1841. The scale of this operation, which is the removal of about 2000 cubic yards (40 yards in front, 3 from back to front, and 16 high), makes it very interesting, and its recent occurrence, as well as that of another great *éboulement* opposite Souillols, throws light upon the natural mode of proceeding. In both these cases the atmosphere seems to have acted alone. At Jaujac (the case represented in Plate VI., fig. 3), the river did not touch the base of the lava at all; we must, therefore, distinguish two methods of disintegration. 1*st*, The atmospheric water penetrating freely amongst the countless fissures which the imperfect columnar structure of the upper bed presents, detaches it gradually by the usual effect of liquid pressure, more rarely perhaps, by congelation. The principal tendency to fracture being exactly in a vertical direction, the cliff has a continual tendency to instability. A few hundred yards from the spot last described, there was, at the time of my visit, another huge mass in the process of separation from the cliff by a gap already two feet wide at the top. I passed below it during a tremendous thunder-storm, when the rain-water was gushing in torrents from the joints of the pillars, threatening instant precipitation. 2*d*, The *lower* part of the lava being always regularly columnar, and having, therefore, very little lateral cohesion, and probably, like a table with many legs, all the pillars not bearing equal shares of the load, the erosive action of the water must necessarily detach them; and, we almost always, if not invariably, find that the upper part of the cliff projects, as in Plate II., fig. 6, and gravity thus assists the atmospheric causes of disintegration.

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* A description of the Western Islands of Scotland, vol. ii.

To these I would add one consideration as to the *commencement* of erosion. When the form of a lava-bed confines the running water to the centre of a stream, as in fig. 7, we find that it acts extremely slowly;* but if the river takes *one side* of a lava-bed, as in fig. 8, there being a crevice between the granite and the lava, the water must penetrate, and, by its pressure, tend to separate the columns, and to wash them out, so that I know of no *existing* case of water running under the condition last described. But *portions* of the lava often adhere to the granite even on the side of the valley most eroded. This is the case on the bank of the Alignon.

The often excessive fragility of the lava also assists its division and removal. At the *éboulement* of Jaujac the basalt is singularly brittle, almost the whole fallen mass is shattered into bits of a few pounds weight. In texture it slightly resembles pitch-stone, and the lustre is that of animal glue. *The fragments include many pieces of granite.* At the junction of the Alignon and Rioclat is a considerable mass of pure feldspathic granite, not like that of the country, surrounded by, and cased in basaltic columns, which have formed almost as regularly as if it had not been there. This, though very interesting, is conformable to the illustration which we gave of the law of direction of the columns. If it depended upon the contact of the lava with the granite otherwise than as the cooling is thereby affected, the columns might be expected to radiate from the enclosed mass; but as any mass, not enormous, enveloped in such a stream, might acquire the temperature of the melted matter, the whole would cool with reference only to the conditions extraneous to it. The lava extends but little way up the Rioclat, which, however, presents a curious deposit of volcanic ashes, which no doubt must have fallen from the crater of Jaujac.

The *Coupe* de Jaujac (being an exact translation of the word *crater*, by which the ancients denoted a volcanic orifice) is distant only half-a-mile or a mile from the village, in a natural opening or cavity between two primitive mountains, filled with coal-formation sandstone, whose character is well marked by abundant impressions of fossil plants. This formation (in which coal was at one time worked)† is interesting on several accounts, although the occurrence of the volcano in the midst of it cannot but be regarded as entirely accidental. Its extent is small, being nearly confined to the valley of Prades, adjoining that of the Alignon, at least if the Geological Map of France be correct. It is, as already observed, part of a widely-spread but often-interrupted ring of the coal-formation, surrounding the primitive *plateau* of Central France. I examined, very carefully, the whole exterior of the volcanic cone, and found the sandstone *everywhere*, except in the narrow space occupied by the issue of the lava-stream from the crater (see Plate II., fig. 5), which abuts against a hillock of sandstone, marked C on the plan. It does not, however, appear to reach the banks of the Alignon, and terminates abruptly against the two hills of gneiss, A and B, which form part of the boundary of the valley, which is everywhere of primitive rock. Near A, the junction of the coal-formation and the gneiss is well seen. The strata of sandstone and gneiss resemble each other so much, that, being nearly conformable, it was some time before I could assure myself that the fine granite veins which occur there were not in the coal-formation.

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* *Examples:*—The tributary on the left of Ardèche near Thuez; lava of Burzet at the cascade near the village.

† On the authority of FAUJAS ST FOND.

The crater is probably the largest in the Vivarais. It is low, strong, and of an elliptical form, and has burst at one end of the longer axis, being that towards the village of Jaujac. From the firmness and dimension of the lava walls, I presume that it must have been lofty, as is indeed probable, if it contained any large part of the prodigious flow of lava which proceeds from it, and which is evidently the result of one eruption, and probably the only one which this volcano has experienced. The interior of the crater is filled with clay and ashes, which sustain, however, a magnificent growth of chestnut trees. The open lip is narrow, and just gives vent to a stream of fine compact lava with little slag, which then fills the valley; the town of Jaujac stands on the extreme upper part of it. Near the mouth of the crater, and 1531 feet above the sea, is a spring plentifully charged with carbonic acid, whose temperature was 53°·2. A spring, issuing from below the basalt, at the junction of the Alignon and Rioclat, had a temperature of 54°·7 (23d June 1841). These temperatures are both above the mean of the place. The extreme height of the *Coupe* (by the mean of barometrical observations in 1839 and 1841) is 1923 English feet above the sea. The level of the surface of the basaltic formation at Jaujac is about 1400 feet; and the top of the *coulée* of Neyrac, at the junction of the Ardêche and Alignon, is 1117 feet.

Valley of the Ardêche—Thuez.

If, instead of pursuing the course of the Alignon to Jaujac, we follow the principal stream of the Ardêche to Thuez, we first of all seem to leave all traces of basalts in the valley. Very soon, however, patches of volcanic formation appear upon the right of the road; whilst, on our left beyond the river, we have the volcano of Neyrac or Souillols, formerly mentioned as occupying the ridge between the two valleys, and as having unquestionably thrown out its principal stream into that of the Alignon. The summit of the volcano (which is at a height of 2178 feet above the sea, barometrically determined in 1839) commands a very fine view of the upper district of the Vivarais, and of the volcanoes which

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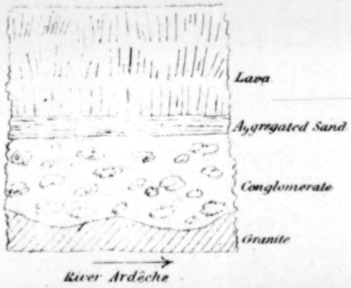
Fig. 1 p. 17 Section at Neyrac



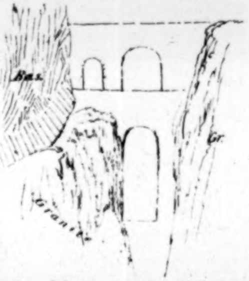
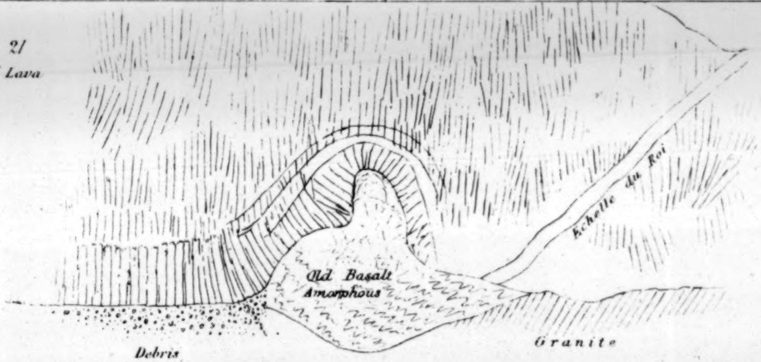


Fig. 2 p. 20 La Gueule d' Enfer _ Thuez

Fig. 3 p. 21

Superposition of Lava near Thuez



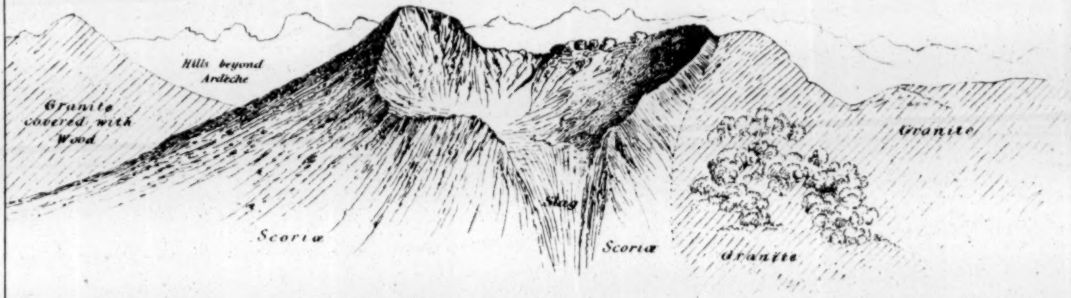


Fig. 4 p. 22 Gravenne of Montpezat from the North

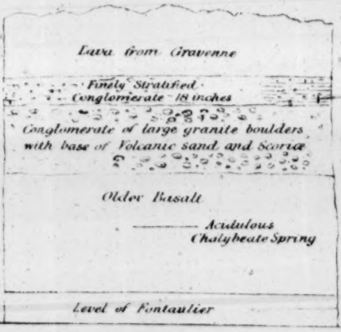
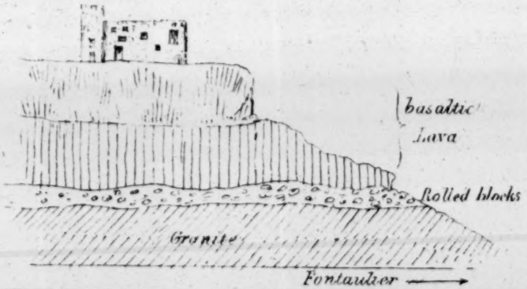

Lava from Gravenne
Finely Stratified Conglomerate 18 inches
Conglomerate of large granite boulders with base of Volcanic sand and Scoriæ
Older Basalt
Acidulous Chalybeate Spring
Level of Fontaulier

Fig. 5 p. 23

Section at Bridge of Aulière near Montpezat

Fig. 6 p. 23 Castle of Pourchirol



W & A K Johnston lith. Edin.

occur there, as well as of the valley of Thuez. It has a distinct, though degraded crater, which has given way on the south-eastern side. The sides of the ridge are wooded; but I traced two distinct streams, though of no great dimensions, into the valley of the Ardêche, which evidently came from the crater of Neyrac. They include between them the hamlet which bears that name; and one at least of them (the most westerly) may be traced down the bed of an insignificant watercourse in the granite, down to the valley, where it has formed columnar basalt for a short distance. The other patches of basalt, between this point and the junction of the Alignon, are probably due to this stream.

Opposite to the village of Neyrac, the following section occurs (Plate III., fig. 1), which attracted my attention from the complete state of aggregation of the sand and gravel intervening between the basalt and the granite soil. It is evidently formed by the concreting action of calcareous matter, held in solution by the carbonic acid which occurs abundantly in the neighbourhood upon the detritus of the valley, which it has compacted to such a degree as to form a kind of sandstone. The carbonic acid occurs in a spring rising in a neighbouring meadow, and having a temperature of no less than 78°·5, whilst a small spring, between it and the river, marked only 51°. The elevation of the mineral spring above the sea is 1359 feet. In the side of a granite hill, close to the hamlet of Neyrac, is a dry discharge of carbonic acid gas, producing a suffocating atmosphere, similar to that of the *Grotto del Cane* near Naples, the effects of which have been elaborately described by Faujas and other older writers on the Vivarais.

Continuing to ascend the valley from Neyrac, either by the road or by the river, objects of interest multiply. The slopes on the right are occupied by an immense mass of red cinders, and slaggy lava and ejected bombs, indicating the close proximity of a crater, round the outside of which, indeed, the road winds for some distance,—whilst the valley seems absolutely barred in advance by immense cliffs of basalt, which tower over rugged rocks of granite, amongst which the river struggles. Across it has been thrown an extremely slender and elegant arch, which bears the name of Pont du Diable, or Pont d'Enfer. The basaltic cliffs rise to a height of nearly 300 feet above the level of the stream, and upon the plateau which they form is the mean but charmingly-situated village of Thuez. For variety of outline, luxuriance of vegetation, rich colouring, and romantic forms of ground, few spots can be compared to it; and, on my last visit, disregarding the very indifferent accommodation which it affords, I made it the principal centre of my excursions. A short description will, I hope, tend to give a distinct idea of its situation, and the chief points of picturesque as well as of geological interest.

The reader must imagine the bed of a rapid stream (the Ardêche) to have been worked out through the lapse of ages, by natural operations, to a great depth in a soil of granite,—that near the junction with a tributary stream on the left, a powerful volcano suddenly opened, emitting a torrent of lava which filled the greater part both of the lateral and principal valley, damming back the river Ardêche so as to form a lake, which was only drained when the unceasing and all-powerful action of water had excavated a channel in the hard basalts, and even in the granites beneath them, leaving a cliff, which has been already described, towering above the walls of primitive rock, between which the stream struggles, forming a scene far more varied and picturesque, though less regular, than the columnade of Jaujac which we have lately quitted.

Dire must have been the confusion which the element of fire wrought in this quiet valley. Scarcely less appalling the confusion and organic change produced by the subsequent action of the water upon the intruded masses which had for a while staid its course. But now all is again tranquil, and the progress of events is marked by almost imperceptible gradation; for now, the lava being nowhere in contact with the stream, the action of the river on the granite may be considered as inappreciable. The platform raised by the lava, and terminated by a tremendous chasm on the side next the valley, is covered by a most luxuriant vegetation. The village, surrounded by its vines and maize, is exactly opposite to the spot whence the volcano of Mouleyres or Thuez must have evacuated the most abundant part of the fiery flood which charged it; the crater itself, which rises to 2026 feet above the sea, is filled (as usual) with splendid chestnut trees, and its porous cone is planted, like the sides of Vesuvius, with productive vineyards; the site of the lake once formed by the waters of the Ardêche is now a fertile meadow; and the chasm once so ruined and bare, in which the river flows past Thuez, is now ornamented with luxuriant wood of walnut, chestnut, and beech, which give a great charm to the contrasted outlines of the jagged granites, and the alternately level and perpendicular basalts. In many places the *primitive* soil of the valley, the granite surface, cleared of the prodigious load of black rock which for ages covered it, is again brought into cultivation and yields abundant crops.

I have attempted to give (in Plate VI., fig. 5) a panoramic sketch of the position of Thuez, which may serve to illustrate this description. It is taken from the granitic heights on the opposite side of the Ardêche. In the centre of the view, in the middle distance, is the village of Thuez, resting, as has been explained, on the almost horizontal plateau of basalt, whose front or section is presented to the spectator, and which stretches for a great way to the left, or *up* the valley of the Ardêche, with gradually diminishing elevation. In the nearer part of the drawing, on the left, is seen the deep chasm *in the granite* rocks through which the river passes. Immediately behind Thuez, at the closed end of the little lateral valley which we have mentioned, rises the *Gravenne of Montpezat*, a volcano which has thrown by far the greater part, if not all, of its lava in a contrary direction, and which, therefore, has not produced at least any sensible portion of the basalts of Thuez. It is to the open and degraded crater on the right, whose vivid red colour contrasts splendidly in nature with the bright green of the trees around and within it, that we are to look for the source of the great eruption which has at one time levelled the whole valley. A small ravine and bridge may be observed between the volcano and the village. This is called the *Gueule d'Enfer* by FAUJAS, and is figured by him; the Pont du Diable is concealed by the foreground of the drawing. The Gueule d'Enfer now affords passage to the stream belonging to the lateral valley. It is almost entirely formed of granite, and the contrast of the granite and basalt may be traced on the left of the bridge by which the ascending road from Aubenas reaches Thuez.

Now, in the first place, as to the *origin* of the lava there can be no doubt. The volcano of Mouleyres or Thuez has evidently produced at one prodigious throe all this mass, and may be said to have expired in the effort; for there is no appearance of any repetition of the action, and the crater is burst completely open on the side of Thuez, so as to represent, like many volcanic cones near Clermont, the figure of an arm-chair. A portion of the lava seems, however, to have escaped from the crater towards the south, and to have formed one of the basaltic plateaux in the lower part of the valley, and at a much lower level. The highest part of the crater of Mouleyres or Thuez is 2026 feet above the sea, or 500 feet above the general level of the lava-bed on which the village is placed.*

We shall call the *lateral valley* of Thuez that which is seen in the panoramic view extending towards the Gravenne of Montpezat. I have repeatedly examined the whole of this valley with the greatest care, in order to decide whether the last-named volcano had any share in producing the lava which fills it (as some authors have supposed). There is certainly an obscure appearance of a slaggy lava-stream having descended the Gravenne on the side of Thuez, but it is everywhere covered with loose cinders, whose boundary with the granite may be traced right and left. These cinders are in contact with the slag of the volcano of Mouleyres to the east, and they are lost in the bottom of the valley amongst the multitude of granite blocks by which it is entirely choked, and through which the rivulet (the same which passes through the Gueule d'Enfer) makes its way without leaving a trace of a section which should decide whether or not there is a lava-stream beneath. Even the undoubted lava of Mouleyres can be traced but a very little way above Thuez, in the bed of the stream. It is important to mention that amidst the granite blocks many are found evidently altered by heat; they are heavy, red, and friable, and have no doubt been ejected from the Gravenne. Similar blocks are likewise common at Jaujac and elsewhere.

If we would now trace more accurately the composition and dimension of the lava-stream, as shewn in the valley of the Ardêche, the best way is to descend by a rough footway beneath the bridge across the Gueule d'Enfer. Here we are on the exact boundary of the immense lava *plateau* and the granite to the east,—that is, the granite is under our feet and to our left in descending, the lava is above us and to the right. The remarkable section (Plate III., fig. 2) of the Gueule d'Enfer shews plainly that the lava *must* have had a support which piled it up, when fluid, to the level which it still retains, and the position of the barrier is conclusively shewn by the direction of the stratum of basaltic columns, whose axes are as usual perpendicular to its surface, and which point out with mathematical accuracy the figure of the retaining wall now removed, and replaced by the deep ravine on the right of the spectator, who looks *up* the defile as in fig. 2. We are therefore compelled not only to admit the excavation of the Gueule d'Enfer *since* the lava was consolidated, but we must suppose that a barrier of some kind stretched across the valley of the Ardêche itself, in order to retain the prodigious lava flow at the great elevation which it has attained, and which causes its bared cliffs now to overhang the valley to a height of 250 feet, reckoning from the bed of the stream. A careful examination of the panoramic view will clearly prove the surprising dilemma in which we are placed. The almost perfect horizontality of the whole remaining surface of the lava proves that it consolidated tranquilly at that level; and yet we find to the right nothing but a wide open valley, which presents no trace of a support, and from which the lava itself *has totally vanished;* for the most scrupulous examination of the bed of the Ardêche has shewn me that there is not a volcanic vestige in its neighbourhood so far down as the environs of Neyrac; and, though it is as plain as the truth of hydrostatics that the basalt must have once filled up the whole bed of the Ardêche at this place, and abutted against the granite-hill opposite (from whence the panoramic view in Plate VI. is taken), there is now not a trace of it on the southern side of the river. How astonishing, then, must have been the excavating power which has not merely disintegrated the mass of lava which has disappeared, but has destroyed the barrier, by means of which it was accumulated to the level which it retains! It is certainly conceivable that this barrier might have been partly composed of the dejections of the volcano which, when much higher than at present, may possibly have extended its cone so as partly to close the valley; yet the whole circumstances appear to shew that the forms of these volcanoes have not materially changed since the completion of their eruptions, and that certainly no vast or powerful streams of water, sweeping over the whole country after the manner of a *debâcle*, can be invoked in explanation of the last excavation of the valleys; for the loose texture of the ashes, which repose upon every volcanic cone, would have given way at once before the action of a flood, however gentle. This argument has been effectively used by Mr SCROPE, to prove that the removal of the lava beds can be ascribed only to the action of water following the channels of the present rivers; and has been enforced by Sir C. LYELL and Sir R. I. MURCHISON, in a paper published many years ago, on this very subject,* in which the excavation of the granite, as well as of the lava of Thuez, is cited in additional confirmation. But none of these authors, so far as I recollect, have referred to the singular manner in which this lake of lava has been, as it were, suspended in the middle of a valley which presents so great a declivity. The section just given in the Gueule d'Enfer seems to shew that that ravine must have been entirely excavated since the lava was consolidated! There are few phenomena, geologically so recent, which appear more unaccountable, more disproportioned to the means by which apparently they must have been produced. The facts before us recal, in a striking manner, the *parellel roads of Glen Roy* in Scotland; lake terraces apparently of a similar age to the basalts of the Vivarais (that is, posterior to many of the ordinary river alluvia), and which must have required barriers far exceeding in dimension those which dammed up the lava of Thuez. But in Glen Roy, whatever were the barriers, they were certainly not composed of *solid rock*. Here, on the contrary, they would appear at least in a great measure to have been so.

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* The elevations in this part of the valley were deduced from barometrical observations chiefly made in 1841, and referred to Thuez as a standard height. This latter has been estimated at 1545 feet above the sea at Marseilles from six observations in 1839 and 1841, compared with those of M. Valz at Marseilles, and kindly communicated to me by that excellent observer.

As we issue from the *Gueule d'Enfer*, we find a tolerably wide and cultivated ravine, entirely based on the primitive rock; whilst the mural precipice of lava, in some places 200 feet high, extends for about a mile parallel to the river on the right. The contact of the lava with the ancient soil may almost everywhere be traced. The lower part is usually composed of vertical columns; the upper part is (as at Jaujac) only very imperfectly prismatic; but the whole is evidently the result of one eruption. The lower part of the stream, where it touches the soil, has in some places a very singular appearance, glistening and coaly; probably composed of nearly pure augite mixed with carbonaceous matter, of which I found a singular proof in a portion of a vegetable stem, of which a *cast* has been made by the lava.†

The general phenomenon of the perpendicularity of the lava prisms to the surface of cooling is everywhere exemplified; but nowhere so beautifully as near the part of the cliff called the *échelle du Roi* (from a narrow steep passage formed by a dyke or vein in the lava, and by which the cliff may be ascended). Here there occurs beneath the prismatic lava a shapeless mass, apparently of old lava, slaggy, and not at all columnar; of which the new lava has formed an exact cast, and fringed it all round with columns perpendicular to its own most irregular surface. This is represented in Plate III. fig. 3, from a rather careful drawing made on the spot. I have often been inclined to think that the old lava, which must evidently have been cold when it was overflowed by the other, may be derived from the Gravenne of Montpezat; and this idea is confirmed by the consideration, that this point was undoubtedly the ancient issue of the stream of the lateral valley which now escapes at the Gueule d'Enfer. I have ascertained by barometrical observations that this is really the lowest point of contact of the lava with the ancient surface of the granite valley; and was therefore the *thalweg* or water-way until it was choked by the lava of Mouleyres. If, then, the Gravenne gave birth to any stream of lava, however small, it might be expected to have flowed in this direction. If this change of the water-way be admitted (and it appears to be unquestionable), there is another proof, amounting almost to demonstration, that the Gueule d'Enfer has been excavated in the granite or gneiss since the lava of Mouleyres flowed; for before that time no water could have run through it.

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* Jameson's Edinburgh Philosophical Journal, 1829.

† The specimen illustrating this curious fact, and others referred to in this paper, are now placed in the Museum of the Royal Society of Edinburgh.

The following are the results of my barometrical observations:—

	Eng. Ft. above Sea.
Summit of Lava Cliff at Echelle du Roi, ...	1476
Foot of	1283
Summit of Lava Cliff at Gueule d'Enfer, . . .	1514
Foot of	1343
Height of Cliff at Echelle du Roi (A), ...	193
... ... Gueule d'Enfer (B), ...	171
Surface of Lava at A *below* Surface at B, ...	38
Contact of Lava with Ground at A *below* contact at B,	60

The lava of Thuez gradually thins out as we ascend the valley; continuing, however, to present a basaltic cliff towards the stream. Where the lava ceases, the valley expands; and the river has a wide bed formed of detritus, preserving a uniform level, evidently occasioned by a lake which once existed, due to the obstruction caused by the lava to the waters of the Ardêche.*

Montpezat.

A short walk from Thuez leads to the top of the volcano called *La Gravenne de Montpezat;* but the ascent is rapid, the strata of pozzuolana constituting the southern face of the Gravenne dip at an angle of about 30°, the superficial debris 25°. The principal discharge of lava being towards the north, filling the valley of Montpezat, we shall next in order consider the phenomena which that district, watered by the Fontaulier, presents. The Gravenne itself rises to a height of 2727 feet. The view from it is extensive and striking. It has an exceedingly well-formed crater, which, however, is seen from but few points. Plate III., fig. 4, shews its appearance, as seen from a hill to the north, on a ridge between Mont-

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* About 5 hours' walk across the hills to the south-west of Mayres, or 7 hours' from Thuez (passing le Chambon and Bornes), are the remote mineral-springs of St Laurent les Bains, having a temperature of 125° Fahr. They rise from mica slate, in the neighbourhood of granite, and traversed by granite veins. They contain salts of soda almost exclusively, and particularly the carbonate.

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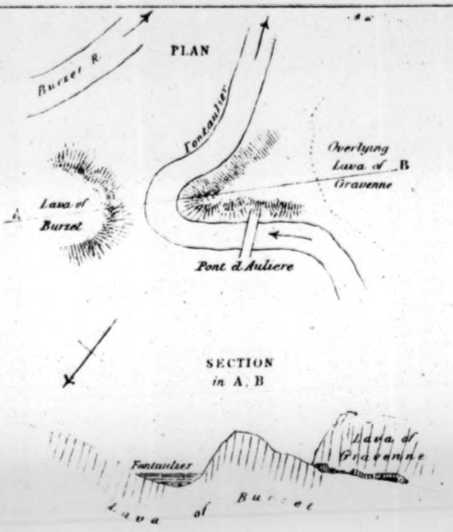


Fig 1 p. 23
Superposition of
Lavas in Valley of
Montpezat.

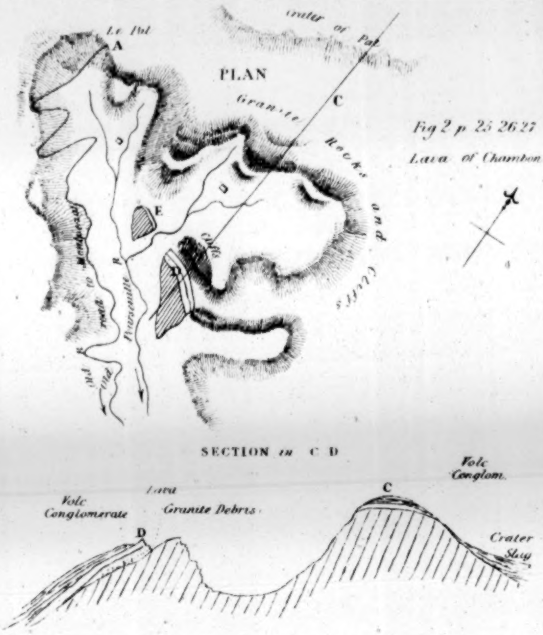


Fig 2 p. 25 26 27
Lava of Chambon

Fig. 3 p 27 Elevation of the Volcanic formation of Chambon

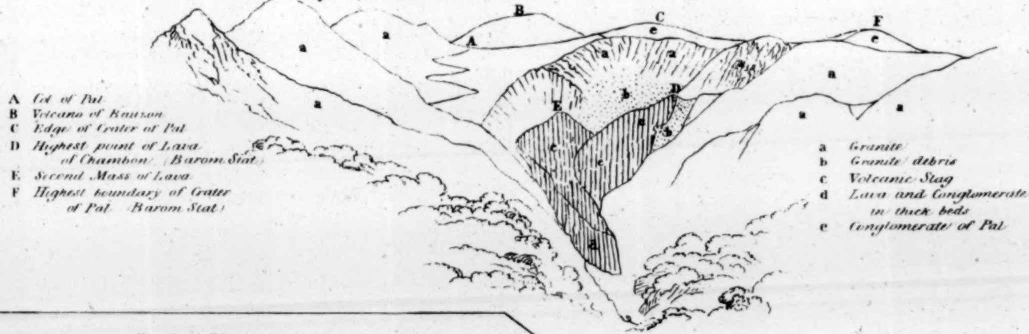

B
C
A
F
a
e
b
D
E
c
d
A Col of Pal
B Volcano of Bauzon
C Edge of Crater of Pal
D Highest point of Lava of Chambon (Barom Stat)
E Second Mass of Lava
F Highest boundary of Crater of Pal (Barom Stat)
a Granite
b Granite debris
c Volcanic Slag
d Lava and Conglomerate in thick beds
e Conglomerate of Pal

Fig 4 p 28 Crater of Pal

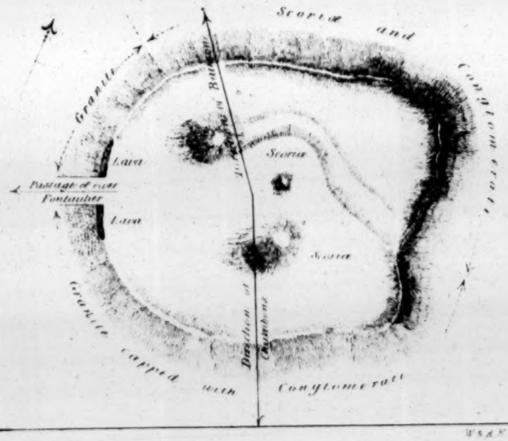
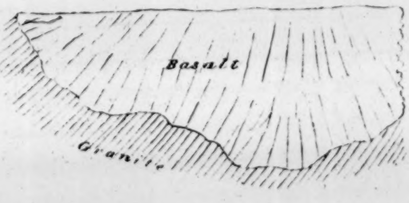


Fig 5 Near Burzet p. 29



pezat and Burzet. It is a real *cup*—the bottom being lower than the lowest edge of the crater by which the lava has poured out into the valley. A high and solid conoïdal mass of lava and ashes commands the crater to the SW. so as to give the character of an unbroken top to the hill in that direction. The highest parts are composed of partly solid lava, and rise 100 feet above the bottom of the crater.* The principal lava-flow descends in the direction of Montpezat, at an angle (by estimation) of little less than 35°; yet it is tolerably continuous. At its base, near the bridge across the Fontaulier (which is 1534 feet above the sea, and 1200 below the summit of the Gravenne), we have a remarkable section, Plate III., fig. 5, which is made *transversely* to the direction of the lava stream of the Gravenne, which forms the upper layer. The layer of basalt forming the base of the section, and which is separated from the other by a layer of granite boulders, cemented like that of Neyrac by the action of the acidulous water, is evidently the product of an altogether anterior eruption. The more modern eruption has bathed the whole valley with lava, which has formed a tolerably level prismatic bed, stretching some way up the tributary stream of the Pourseuille, as well as up the valley of the Fontaulier, but not far in either case, and stopping considerably short of the town or village of Montpezat, which is built upon a mass of granite boulders. The junction of the Pourseuille and Fontaulier affords an excellent section below the picturesque site of the Castle of Pourchirol, Plate III., fig. 6. Here but one bed of basalt appears (I mean due to one eruption).

The plateau formed in the valley of the Fontaulier by the lava of the Gravenne is justly described by Mr Scrope as having "its upper surface bristled with rocky and scoriform projections; which, however, by decomposition, resolve themselves into a rich soil, affording nourishment to very productive chestnut forests." Many scenes of great beauty occur on this plateau. There is some appearance of a lake having been formed at the contraction of the valley at the Pont de la Motte, several miles farther down; and both in this valley and that of Burzet (which we shall afterwards describe) there occur deposits of pozzuolana apparently stratified by water. At the junction of the Fontaulier with the tributary valley of Burzet, about half-way between the Gravenne and the Pont de la Beaume, the lava of Montpezat joins that of Burzet, or rather, overlies it,—the plan and section, Plate IV., fig. 1, marking plainly the more recent date of the eruption of the Gravenne. These streams remain superposed for some distance; but at the Pont de la Motte (at Mayras) there is only a single stream, which is columnar, abounds in olivine, and reposes on a conglomerate. I am unable to state positively to which of the two valleys this may be traced (although the determination would be easy); but from its composition and other circumstances, I am inclined to think that it derived from the valley of Burzet. It continues all the way to the *Pont de la Beaume*, and is no doubt the same which we have already described as underlying the great *coulée* of the valley of the Ardêche.

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* Messrs Lyell & Murchison (Edin. Phil. Journal, 1829, p. 27) speak of angular blocks of unaltered gneiss as occurring near the summit of the Gravenne. These appear to have escaped my notice.

Having now traced the course of the lava of the Gravenne, we shall point out some other volcanic appearances in the valley of Montpezat. And, first, we have a dyke of basalt, in the granite of the left bank of the Fontaulier, near the hamlet of Les Plantas, about half-way between the Castle of Pourchirol and the entrance of the valley of Burzet. This is a phenomenon which we should naturally expect to meet with very frequently in a country which, with a soil entirely primitive, has been pierced at so many points within a short space by volcanic ducts, which can hardly have been formed without an intense pressure from below, which might have been expected to rend the strata of gneiss in all directions, and then to have filled the rents with melted matter, constituting true dykes, such as those which occur in Monte Somma, near Naples. The reverse is, however, the case: and the dyke of *Les Plantas* is the *only* one which I have met with in *this part* of the Vivarais (amongst the older volcanic formations of Le Puy and the Coyrons they are numerous). No doubt, many may have escaped me; and I should never have known of this one, owing to its remote and concealed position, had I not been fortunately directed to it by a resident at Montpezat. It occurs in a small ravine (see the map), running in a direction nearly north and south, and is said to have been traced for a mile. Its plane is vertical. Its breadth varies from 1 to 4 feet; and it sends forth small veins into the rock, and includes portions of granite in its substance. In composition, it resembles the lava of the district; black in colour, with concretions of olivine.

The village of Montpezat, charmingly situated on the rivulet Pourseuille, at the height of 1857 feet above the sea, does not stand upon the lava of the Gravenne, but about half-a-mile beyond where it terminates at the parish church of St Pierre. The valley is filled by a prodigious multitude of rolled blocks of granite; which here, as in the lateral valley of Thuez, appear to be out of all proportion to the extent of the ravine from which they must have been derived, and of which the countless multitude contrasts with the comparative rarity of basaltic blocks. At the head of the valley, near the source of the Pourseuille, rises a mass of volcanic cinders, well distinguished at a distance by its colour; and which is figured by Mr SCROPE, in his view of the valley of Montpezat (Plate XV. of his work), who describes it as an anonymous volcanic cone which has not produced any lava stream. He had evidently not visited the place; for the true description is very different. I shall now detail the chief results of a patient examination of this locality, both in 1839 and 1841; there is no point in the Vivarais which I have more narrowly investigated.

Montpezat, we have said, is placed upon an immense bed of granite boulders, which extend upwards, occupying the whole bed of the valley, to a great thickness; but near the village of Le Fau we find a mass of lava, which, with other volcanic products, fills the bed of the stream for a very considerable distance, rising with a very steep acclivity, which shews that its viscidity must have been considerable to allow it to harden at so great an inclination. M. Serrouil, watchmaker at Montpezat, showed me a sapphire and zircon which he had picked up near this place, which is interesting from the occurrence of these minerals in the lavas of Croustet, near Le Puy. The lava of Le Fau is surmounted by beds of volcanic conglomerate, of which the river has made a section in passing between it and the granite on the right bank of the valley; the thickness of the conglomerate is here at least 100 feet, and the inclination may amount to 25°. The volcanic stream presents a very massive and striking appearance, and leads to the belief that we are in the immediate neighbourhood of a volcanic focus; and such might be supposed to exist in the steep slopes covered with slag and volcanic conglomerate (the conglomerate still uppermost), which rise to a great height on the left bank of the stream, not far from its source. Nothing of the kind, however, appears; the beds become thinner, and are in absolute contact at no great depth with the granite which constitutes the neighbouring slopes, or with granitic boulders covering the rock at D, Plate IV., fig. 2. The lava in contact with the granite has evidently consolidated in its present position, from the nicety of its adaptation to the rock, and from the *shortness* and fragility of its structure. The position of this volcanic *curtain* will be best understood from the plan in the figure just referred to, which shews the uppermost part of the valley of Montpezat, as traversed by the very steep paved road leading by the hamlet of Le Pal from Usclades and Le Puy. This road has since been superseded by a new one, which was nearly completed at the time of my last visit. It will be seen that the highly-inclined beds D, are those now described, and are divided by a streamlet which issues from a kind little circular valley, in which a cottage is marked.* This circus might well pass, upon a superficial examination, for the source of the lava or a true crater. It has, however, no such pretension, being completely excavated in granite; and the aforesaid curtain of lava and breccia forming a thin exterior facing on the side exposed to the high road. The height which the volcanic formations attain on this slope is no less than 4134 feet above the sea, and nearly 2300 feet above Montpezat, though so little distant. It will be understood, therefore, that the slope is extremely rapid.

The place where these remains occur is called *Le Chambon*: we shall therefore denominate the deposit by the name of the lava of Chambon. It is hoped that it has been made evident that no eruption took place in this spot. The excavation of the ravine behind D, and the numerous contacts with the granitic soil, shew that it is entirely foreign to the place. Whence then is it derived? There can be but one answer, even although that conclusion be attended with some difficulties. It must be a stream from the *Crater of Pal*, already adverted to in describing the route by the Bauzon from Le Puy to Montpezat. This well-formed, nearly circular crater occurs a little behind the granite rocks and cliffs which compose the back-ground of the ravine of Chambon; and so far it would appear to be a very natural origin of the lava. But the singular circumstance is, that the crater in question belongs to a different system of valleys; inasmuch as the water issuing from it forms the source of the Fontaulier, and reaches the vicinity of Montpezat after a long circuit (see the General Map, Plate I.). It is also pretty open to the north; but it seems completely cut off from the ravine of Chambon by a considerable granite ridge running NW. and SE., forming the *Col* or passage o. Le Pal, and rising to a height of 4537 feet, exactly between the crater of Pal and lava of Chambon, being 644 feet higher than the former, and 403 feet above the extreme point of the latter. There are therefore two difficulties to be accounted for; *first*, the passage of a great mass of lava and other volcanic matters over this barrier; *secondly*, the excavation of the ravine of Chambon, and the removal of all the upper part of the lava stream.

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* There is also another cottage more to the left, in a place where, on Cassini's Map, is marked "Lac de Ferrand;" that little lake in reality lying higher up in the direction of the volcano of Bauzon.

With regard to the *first* difficulty, it is important to remark that most of the granite heights between Le Pal and Chambon are covered more or less thickly with a volcanic conglomerate forming horizontal beds (one granite top which has escaped, is uncoloured on the map). This conglomerate descends to the crater of Pal, and stops short abruptly on the face nearest Le Chambon (see section, Plate IV., fig. 2). Consequently, the hills about C (fig. 2, *Plan*) did not probably always form the highest part of the edge of the crater; for the conglomerate is of too compact a character to have been formed by a dejection of volcanic materials merely; and the same cause which permitted the deposition of these conglomerates (such as the heightening of the crater by the formation of a cone which has since disappeared), would permit equally the passage of a part of the fused materials across the ridge into the neighbouring valley. And the fact of the occurrence of abundant conglomerates in both positions, confirms, if it does not render absolutely necessary, this supposition; although I am aware that there is some mineralogical difference in their composition,—that of Chambon being more friable, and more generally composed of volcanic ingredients, containing, however, granitic masses,—that of Pal being *chiefly* of granite boulders, cemented by a very hard volcanic basis. The posteriority of the conglomerate eruption may account (together with the great declivity) for the absence of any trace of the *lava* flow on the granite heights of Le Pal.

The *second* difficulty, that of the excavation of the ravine of Chambon since the lava flowed, must remain, I fear, unanswered, upon any theory. Of the fact there appears to be no reasonable doubt. This (geologically) recent excavation of a perfect mountain of hard granite, at the head of a ravine which possesses no drainage sufficient to procure a powerful current of water, and which is near the culminating point of three pretty extensive mountain-ridges, is merely one of a thousand cases of excavation which most hilly countries exhibit, but of which no geological hypothesis has yet given any satisfactory account. In this case, the difficulty of the explanation is increased by the circumstance, that the excavation has occurred since the deposition of a very recent lava, a lava which has itself covered the boulders due to a previous denudation.

I need hardly say one word of M. Burat's assertion, that the lava of Chambon is an "*épanchement lateral*" from the granite wall of the crater of Pal; in other words, that it has issued by a subterranean orifice from the same internal focus. Such an explanation would be highly satisfactory were it founded on fact. But we have seen that no such orifice exists. Did it exist, it must be at a higher level than the highest point of the volcanic accumulations D (fig. 2); but the lava has there only a few feet of thickness, and beyond, above, and around it, the bare granite is everywhere exposed, the circus extending from E to D, being throughout entire, generally precipitous, and *presenting no fissure.* To this may be added the fact, that the highest part of D is higher than the present level of the interior surface of the crater of Pal, as the following barometrical heights attest.

		Barom. mill.	Ht. above Sea.
1839, June 3.	—Lowest part of Crater of Pal, . . .	661·6	3893 feet.
... ...	Highest Boundary of the Crater to NE. (conglomerate), ...	646·9	4537 ...
... ...	Highest point Lava of Chambon, . .	656·1	4134 ...

The whole relations of this singular formation will, I hope, be made plainer by the elevation, or distant perspective view in Plate IV., fig. 3, carefully drawn from the top of the Gravenne of Montpezat. The same letters of reference are used as far as possible in figs. 2 and 3.

The crater of Le Pal has acquired some celebrity by the discussions to which it has given rise in the Geological Society of France;* having been by some regarded as the type of a *crater of elevation*, as composed entirely of granitic strata elevated so as to present a volcano composed of primitive rock. I cannot at all partake of this view; and indeed can affirm, from the most careful and repeated examination, that it presents no peculiarity which can cause it to be considered as exceptional, unless the insignificant quantity of scoriæ ejected by it (or, at least, which are now visible), which is insufficient to form a true cone like that of the neighbouring volcano of Bauzon. The primitive rocks appeared to me here as elsewhere to be undisturbed;—to have no exterior configuration whatever, indicating that they have been moulded by the pressure of lava in the interior; and certainly not to possess the character of "internal precipices (forming the crater) with gentle external slopes:" the section in fig. 2 shews just the reverse. In fact, the granitic eminences by which the crater of Pal is partly surrounded, are (as already mentioned) the culmination of several ridges which have no approximation to a circular arrangement; and they do *not* form the entire circumference of the crater,[*] but only about one-half of it at its internal base, without allowing for the inequalities which even in that part are often filled up with volcanic slag, and the enormously thick coating of conglomerate above the granite on the eastern slope of the crater. Were we simply to project the points where the granite appears, it would make but a sorry circus. In fact, it may be regarded as a granitic valley choked with scoriæ, and plastered with Roman cement till it forms a nearly circular cavity. The occurrence of a crater in such a position only affords, in my opinion, a fresh proof of the singular perforation of non-volcanic strata (including granite or gneiss in that term), by subterranean explosions which have been so incredibly sudden and violent as to have occurred without any visible general disturbance (like a pistol-shot penetrating a board, to use a comparison which I have somewhere read), which recall the extraordinary crater lakes of the Eyffel, penetrating slaty rocks without deranging their strata, and seldom giving birth to any considerable volcanic stream.

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* Bulletin de la Société Géologique de France. Tom. III. and IV.

Fig. 4 of Plate IV. is taken from a rude eye-sketch of the crater of Pal. The chasm at the western entrance gives exit to the river Fontaulier, which takes its rise here. The temperature of the copious spring is 42°·2, the height above the sea being 3900 feet.[†] The existence of a spring of the natural temperature (for such it appears to be)[‡] in such a position leads to an interesting reflection. Its great bulk, and probably uniform temperature, shew that it rises from a considerable depth; and it cannot be doubted that it must follow the course, or nearly so, by which the lavas amidst which it rises have made their way from the interior of the earth. The source of the spring is therefore in the direction of the volcanic focus. It is evident, then, that the mass of rock at that depth, which was once incandescent, has had time to cool to (sensibly) the temperature of the air. How old, then, must be the date of this geological recent eruption? It would not be difficult to submit it to calculation; but the uncertainty of the date would render numerical results of little value. Every one comprehends that a mountain of granite, with a nucleus of incandescent lava, could not have been *completely* cooled unless in a period of time of very great length.

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[*] I am sorry to say, that the view in M. Burat's interesting work on Central France (Plate VII.), is altogether exaggerated and inexact.

[†] 1839, June 3. Spring 42°·0 (Therm. marked A. 1.). 1841, June 25. Spring 42°·2 (Therm. marked A. 3.). Now the correction of A. 3 is +0°·2. That of A. 1 was 0°·0 in 1838. By a singular coincidence the barometer on these two occasions marked *the same tenth of a millimetre.* The temperature of the air was also within 1° Fahr. of being the same both days.

[‡] The mean temperature of Viviers on the Rhone, which is in the same Department, and only 57 mètres above the sea, is 55°·25 according to Cotte, as given in Dove's Tables. The crater of Pal is at 1186 mètres above the sea, or 1129 above Viviers. Now, in France generally (according to Martins), the decrease of temperature with height is 1° cent. for 180 mètres, or 1° Fahr. for 100 mètres exactly; or 11°·29 for 1129 mètres, whence the mean temperature of Pal should be 55°·25 −11°·29=43°·96. The temperature of the spring was somewhat lower in the month of June, but the approximation is a fair one.

There are two portions of slaggy lava near the artificial dam, at the exit of the stream from the crater. But I could in no way identify them with the *dykes* so prominently alluded to by BURAT; but these may possibly have been concealed by the artificial erection. Indeed, I saw nowhere any appearance of injected lavas into fissures in the granite. The crater is stated by BURAT to be 1200 metres (two-thirds of an English mile) in its longer diameter. It is sensibly oval and tolerably flat, presenting three mounds of scoriæ in the centre, similar to those which occur in the craters of recent volcanoes.

Only a few hundred yards in a north-west direction from the crater of Pal is the little *Lac de Ferrand;* erroneously placed in CASSINI's map at the head of the valley of Montpezat. Adjacent to it is a small lava stream (indicated in the map), which might lead us to conclude that this lake had been a small volcanic orifice; this is, however, uncertain. Farther on in the same direction lies the volcano of Bauzon, which has ejected a vast quantity of scoriæ, and is the last of the recent volcanoes of the Vivarais on this side. Otherwise it is without interest. Its height above the sea, by my observations in 1841, is 4922 feet.

Valley of Burzet—Crater of Fiollonge—Valleys of Antraigues and La Bastide—Pic de l'Etoile—Coupe d'Aysac.

We now return to the valley of Burzet, which is a tributary of the main valley of Montpezat, but which extends itself to a great distance in one of the least frequented parts of the country. We have seen (p. 23) that its bed is filled with lava, which was certainly older than that of the Gravenne (see Plate IV., fig. 1). This lava may be traced, with little intermission, as far as the village of Burzet, between 4 and 5 miles from its outlet. The valley is beautifully varied in scenery, and in some places richly wooded. At Burzet it widens, receiving a tributary on the right bank of the principal stream. This tributary takes its origin near the crater of Pal; but its course presents nothing very interesting. The lava widens at the junction of the two streams, and the village is partly built upon it. It would be difficult to select a more exquisite picture of rich and peaceful scenery than the neighbourhood of Burzet. The stream is small, and its bed generally narrow; yet though the lava masses become more insulated and smaller as we ascend, nothing indicates that we have arrived at their commencement. We fancy them to have ceased entirely, when in some re-entering angle of the valley their black abrupt faces reappear, clinging to the granitic soil, moulded upon it, and the columns always perpendicular to the surface of cooling, as in Plate IV., fig. 5. The slaggy traces of the current often rose so high upon the banks, that I repeatedly thought that I had obtained the volcanic orifice. This, and the *concavity* of the surface of the basalt in many places, as in Plate II., fig. 7, shews the extreme liquidity of the stream, which must have shot through this narrow and tortuous valley with a rapidity truly astonishing, leaving its scum and slag to mark the height which it attained; but the main flood of lava not having time to solidify, was propelled ever onwards, leaving often but a narrow thread in the channel of the stream, by which to trace its passage.

One circumstance struck me very forcibly as regards the mineral character of this lava. It contains most abundant nodules of olivine, which have not in the least the appearance of being formed *in* the lava by a slow crystallizing process; but, on the contrary, the irregularity of their forms is *decidedly fragmentary*, often as if rolled; and from the manifest impurities which they present, I am of opinion that they are nothing but fragments of granite in a peculiar metamorphic condition. This idea first struck me several years before I visited the Vivarais, in an examination of the lavas of Clermont; where I found olivine masses near La Barraque, having at first sight the aspect of granite fragments, but when examined, they appear to be only olivine, with some adhering reddish micaceous and clayey matter.* The olivine of Burzet presents two varieties of a pale yellow green, and of a peculiar orange colour. They have no concretionary structure depending on their form. Similar phenomena are common in different parts of the Vivarais, but less, I think, in lavas which are decidedly columnar. The frequency of apparently ejected masses of almost pure olivine in many volcanic countries, and most especially in Upper Eyffel (as at the crater of Dreiser-Weyer) gives an additional probability to the hypothesis that they are foreign substances expelled in an altered condition. Long after making these remarks, I noticed a passage in Mr Scrope's work on Central France, tending to the same conclusion; namely, that these olivine nodules are altered masses of pinite from the granite.†

In 1839 I pursued the traces of the lava of Burzet about 4 miles beyond the village of that name, or at least 8 from the opening of the valley, without seeing a trace of a crater; and my time did not then allow me to prosecute the search. But it was one of the objects of my second visit in 1841 to resume it; and, accordingly, accompanied by my friend Mr John Mackintosh, I slept at Burzet in very uncomfortable quarters, in order to have the whole day for our excursion, resolving to ascend as high as the great ridge which separates the waters of the Ardêche and of the Erieux. We then readily found the point which I had before reached, where the lava temporarily disappears from the valley, though there is no appearance of a crater; and this disappearance continued longer than I expected, as we scrambled with some difficulty along the bank of the river. I carefully marked here, as elsewhere, every patch of lava upon Cassini's map. The disappearance of the lava in this place is no doubt due to the steepness and narrowness of the channel, which left few points of lodgment for the lava, and which gave an increased force to the eroding power of water.

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* M. de Buch in his early writings (Journal des Mines, XIII., p. 251) notices the olivine of La Barraque, without saying anything as to its origin, except that he considers it an exclusive character of old basalt, as contrasted with lavas, at least in that country.

† Scrope, p. 150. Faujas St Fond speaks of the lava at Beaume containing "petits éclats de granit *en chrysolite*," which seems to indicate a similar opinion. (Recherches, p. 300.) He also describes masses of olivine existing in the basalt of the valley of Burzet, in the meadows below the village of St Pierre Colombier, weighing as much as 30 pounds. This remarkable statement deserves verification. See *Recherches sur les Volcans eteints*, pp. 249 and 312.

We passed successively the hamlets of Lespereyres and Chabron; and after the last we kept in the hollow of the valley, and soon got a sight of a great lava cliff, betokening our approach to the crater. Clambering high above the left bank of the stream (which we crossed above Chabron) to avoid precipices, we at length, after a fatiguing walk of three hours from Burzet, obtained a view of a very remarkable cascade, which descends from near the crater which formed the object of our search, which, from a cottage occupying its centre, I called the Crater of Fiollonge. The cascade is called Raïpis; and the water falls by several steps over a mass of beautifully columnar lava several hundred feet high. These interesting objects more than rewarded our perseverance. They are described, so far as I am aware, by no previous writer, and were probably seen by a geologist then for the first time. Plate VI., fig. 4, gives a view of the cascade.

The crater presents, as usual, an insulated cone of scoriæ, touching the granite in its whole circumference, except on the side next the valley, where a flood of lava has made its escape, damming the lateral stream, and forming the great cascade. In the slag adjoining the cascade are immense masses of imbedded granite; which give additional countenance to the notion of the metamorphic origin of the olivine nodules which so remarkably characterize this lava in all its extent. The columnar formation at the cascade, though far from the most extensive, is probably the most curious and complex in its forms of aggregation of any in the Vivarais.

If we now consider the operations of this volcano, we shall find the extreme length of its lava-produce to be really surprising. Although, by the map, the distance of the crater from Burzet may not appear to be more than 4 or 5 miles, the contortions of the bed of the stream are such, that it is probably double that distance; and, indeed, can hardly be less than 8 miles, since we were 3 hours on the way (steep and rough as it was). To the junction with the lava of the Gravenne, in the valley of Montpezat, is therefore more than 12 miles; and if it be the same lava (as Mr Scrope supposes, and as there is some reason to think) which continues to the Pont de la Beaume, the distance will be 16 miles. Yet the whole valley of Burzet is a ravine so narrow and crooked, that the lava appears like a thread winding through it; and we are amazed that it should not have become solidified by the contact with the cold granite, before it had performed one-fourth of the distance. I apprehend that there is no lava known, ancient or modern, which has formed so attenuated a stream. This is only to be accounted for by its excessive fluidity, of which we have additional evidence, (1.)

in the *washing* of the sides of the valley by the scoriæ to a considerable height on either hand; (2.) from the nicety with which it has filled every angle of this irregular channel; (3.) from its almost total disappearance for several miles from its course, shewing (I apprehend) that it ran over the ground almost without consolidating.

	Feet.
The Height of the Pont de la Beaume above the Sea is	1073
... ... Village of Burzet*	1815
Rise of the Valley in 8 miles,	742
Mean Slope (1 in 57, or)	1° 0′

At the crater of Fiollonge the barometer stood, on the 26th June 1841, at 670·7 millimetres. Its height is 3678 English feet. If we deduct from this 300 feet, as a rough estimate of the height of the crater above the bed of the river of Burzet, at the foot of the cascade of Raïpis, we have for the rise in the upper 8 miles of the valley from the village of Burzet, 1563 feet.

Mean Slope (1 in 27, or) 2° 7′

Even on my first visit in 1839, I was so much struck by the fluidity which this lava stream must have possessed, that I made some experiments during the following winter, with the kind permission and aid of the late Mr Edington of Glasgow, upon the flow of cast-iron in very small and tortuous channels at different slopes. The results are scarcely worthy of minute detail; but they left the impression on my mind that the temperature and the liquidity of the lava of Burzet must have at least equalled that of pure cast-iron, a result which strikingly contrasts with the extreme viscidity of most modern lavas, which only attain a long course, aided by very steep inclinations, or by their great volume, which generates a vast moving power as well as retains the high temperature. I found that in tortuous channels of half-an-inch broad, and under slopes of about 1 in 120, the hottest iron used in ordinary casting solidified before it had run a course of many feet. The form of the current, too, was the same as in the lava; the surface was concave, except near the lower termination of the stream, where it became convex, owing to the pressure from behind, and the accumulation due to the increasing friction occasioned by the crystallization of the iron which appeared to be rather suddenly effected. Near the source a mere *trail* of slag was left (often hollow), and the slag bathed the sides of the channel to some height, as described in the case of the lava.

In making these experiments, I little thought that in a few years I should have occasion to return to the subject of the motion of viscous fluids in narrow channels, in connection with the seemingly opposite subject of glaciers; but, in fact, the forms of my cast-iron models (which I still possess), recall strikingly

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* Barom. 715·6 mm. 26th June 1841.

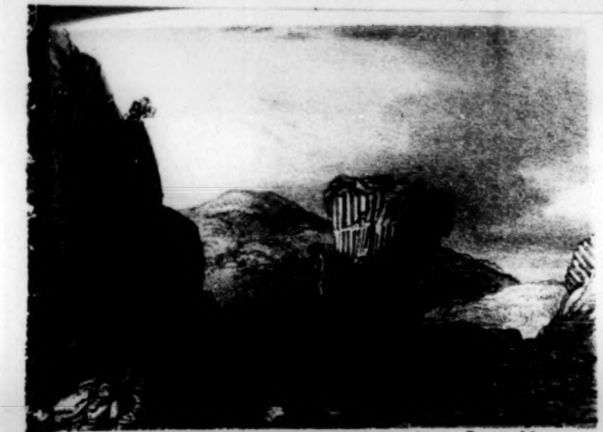


Fig 1 BASALTS AT ANTRAIGUES *Page 33*

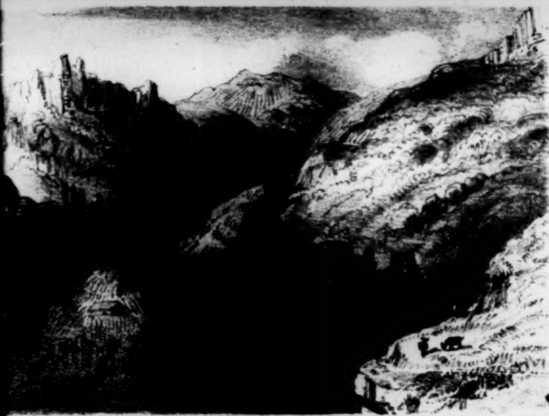


Fig 2 PIC DE L' ETOILE & LAVA OF LA BASTIDE *Page 35*

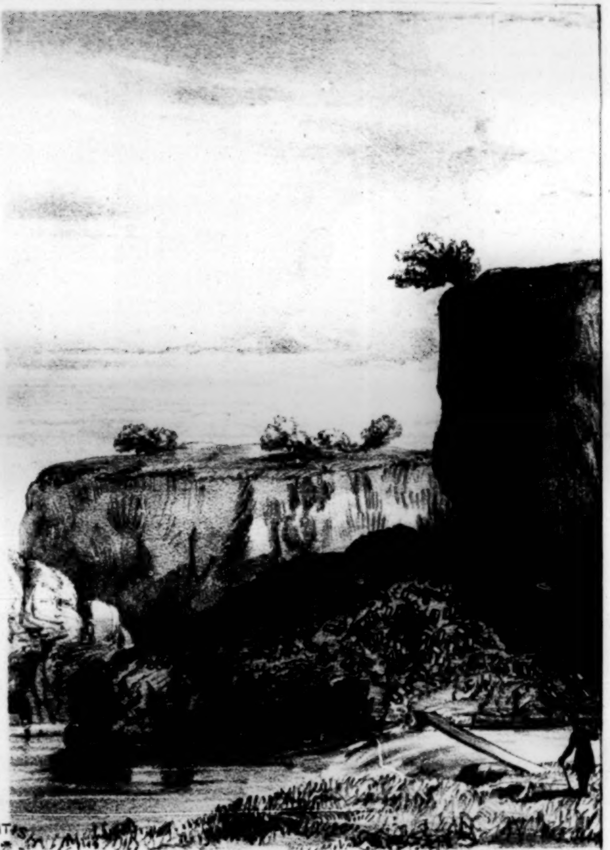
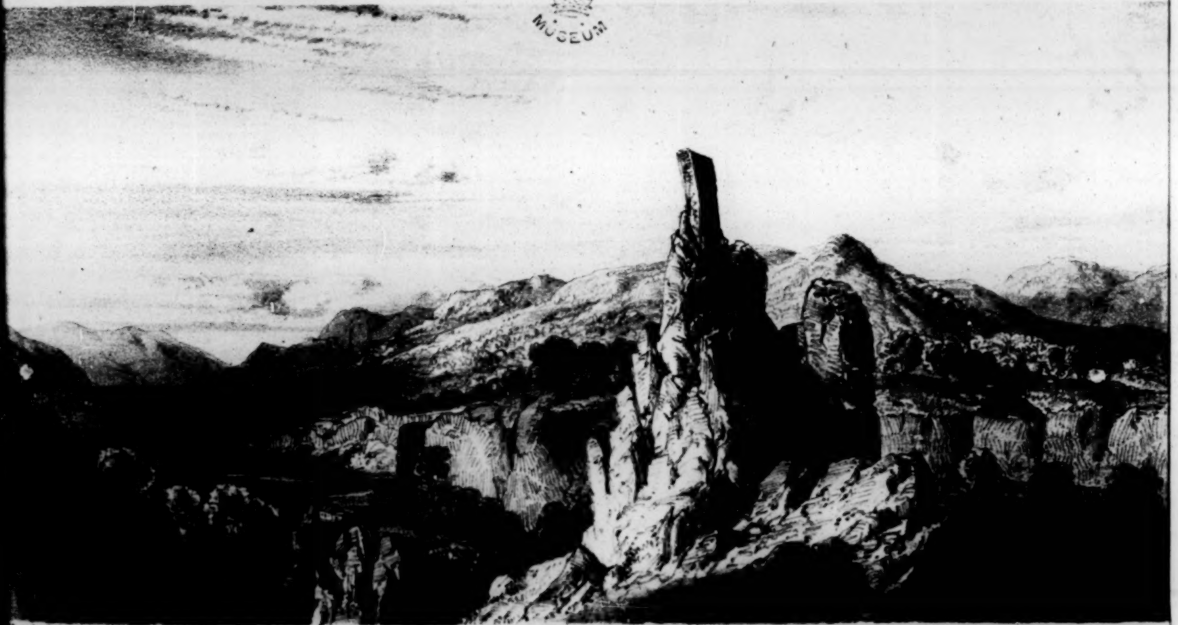


Fig 3 FALL OF LAVA A

BRITISH MUSEUM



ketched from nature by J. D. F. VALLÉE DE MAYRAS.

Fig 5 PANORAMA

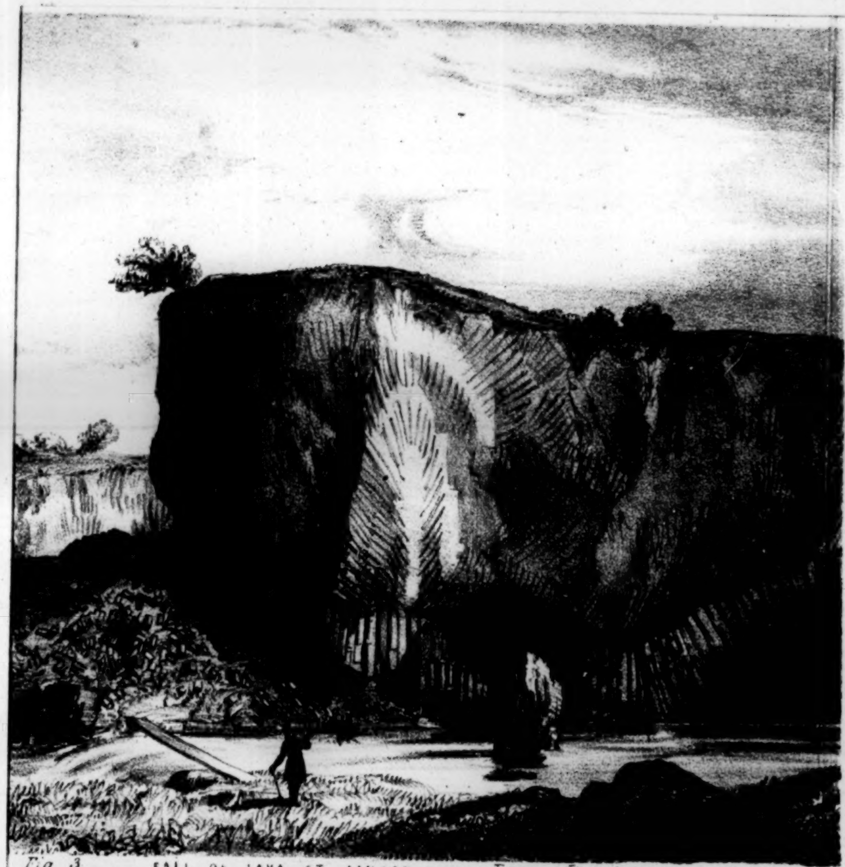


Fig 3 FALL OF LAVA AT JAUJAC. Page 14

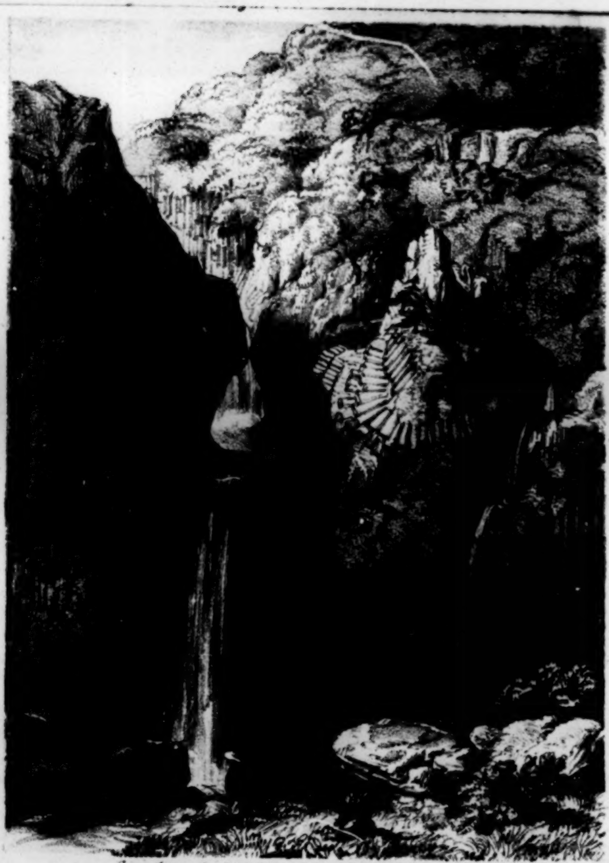
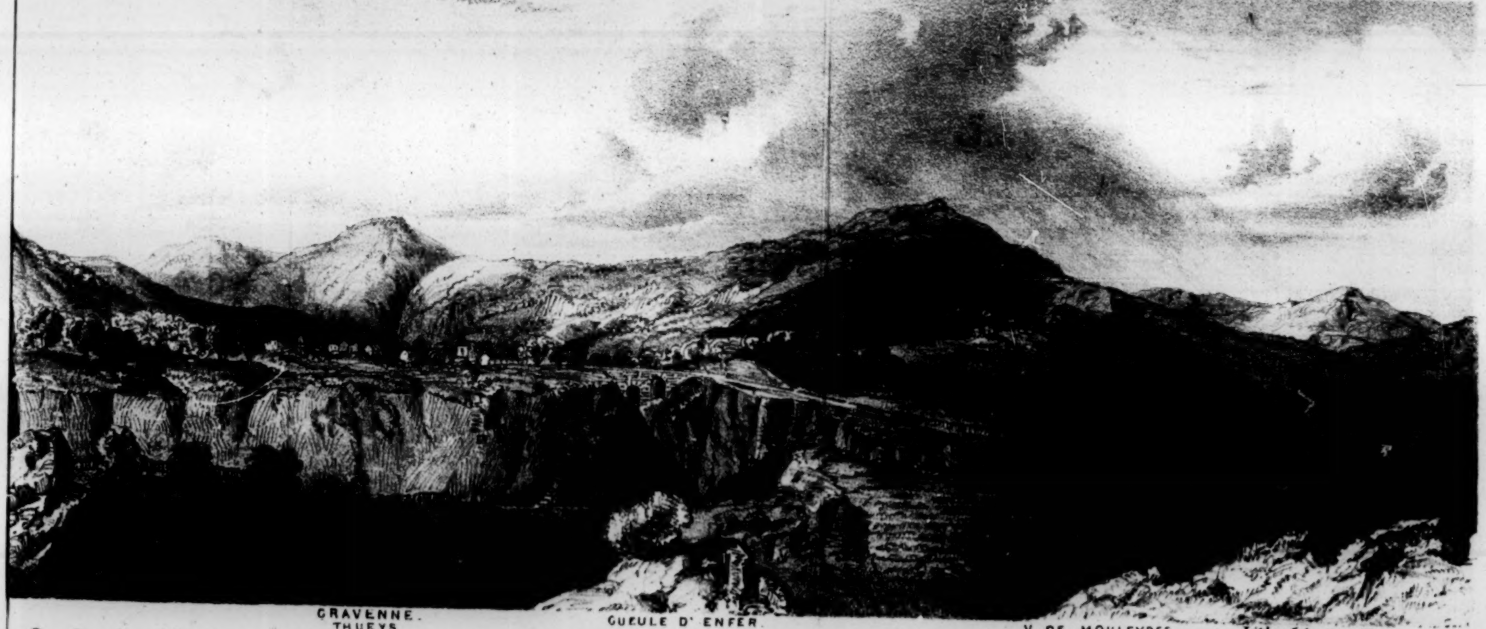


Fig 4 CASCADE OF RAIPIS. Page 31


GRAVENNE.
THUEYS
GUEULE D' ENFER.
V. DE MOULEYRES.

ORAMA OF THUEYS. Page 18

Lithog. by W & A K Johnston Edin.

those of glaciers, both in their higher concave parts, and in the abruptness of their lower terminations.

Figs. 3 and 4 of Plate V. shew the position of the volcanic formation of Fiollonge, relatively to the neighbouring valleys, which are not correctly laid down in Cassini's map.* The way to the village of La † Champ Raphael is by the saw-mill of Bacconnier. The pastures here are filled with charming alpine flowers, and surrounded with extensive woods (*Bois de Cuze*); but La Champ is a miserable exposed village, situated near the heights which overlook St Andeol and the valley of the Erieux, with the chain of the Mezenc in the distance. The elevation above the sea is 4409 feet. In the neighbourhood are old basalts. A few miles to the eastward we reached, by a bleak exposed tract, the village of Mesilhac, where we passed the night in very wretched quarters. The view from hence to the north is similar to that from La Champ Raphael; and I thought the mountain forms sufficiently remarkable to deserve a record, which I have given in Plate V., fig. 5.

Mesilhac is a dirty hamlet of but a few houses, on the *col* connecting the vallies of the Ardêche and Erieux, and is at a height of 3791 feet above the sea. A pleasant walk of only three hours took us next day through a comparatively open and accessible valley to the beautifully situated village of Antraigues, where we found ourselves amidst a wholly new series of modern volcanic formations. Opposite La Viole, a hamlet about five miles above Antraigues, there is a volcanic summit on the right hand of the River Volane; but as its lava is chiefly thrown in a different direction, we shall not at present describe it.

Antraigues (*inter aquas*) is seated, as it name imports, at the junction of several mountain streams in a romantic hollow. It is built on a tongue of granite, formed by the junction of the Volane and Rivière du Mas, and is almost surrounded by patches of a flood of lava, which must once have levelled the whole of this mountain basin. The source of this lava is the beautiful volcano of Aysac. One of the most picturesque remnants of lava bed is shewn in Plate VI., fig. 1.

La Coupe d'Aysac, one of the best-known craters of the Vivarais, is situated almost upon a ridge of granite near the head of a small lateral valley connected with that of the Volane at Antraigues, and pours its lava first into the small valley, and through it into that of the Volane, which has made a remarkable section of it exactly opposite to the village, as accurately figured by Mr Scrope.‡ The lower part, which is basaltic, has, as usual, given way, and left a cavern beneath the little cascade which descends from the lateral valley. The cone of

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* In the View, fig. 4, the hill behind the cottage of La Fiollonge is of granite, but covered with cinders nearly to its summit. The rocks in front on the left are of granite, but those between the spectator and the cottage in the centre, are of lava, as well as in the lower part of the ravine on the right.

† So spelt in the Maps.

‡ Plate XVI. The volume of the River Volane is, however, by mistake, altogether exaggerated.

Aysac is very perfect, with a nearly circular crater, which rises to a height of 2640 feet above the sea, and 1257 above the bridge across the Volane at Antraigues. The crater is extremely well formed, and I estimated its circumference at about a mile. It is broken down at one point only, on the NW. side, by which the lava stream has run down the steep face of the hill opposite to the hamlet of Aysac, where it enters the bed of the torrent, forming a prismatic mass exactly at the foot of the cone. It is from this point that the view of La Coupe, in FAUJAS's work, is taken, and the general features are exact enough. It is therefore without reason that Mr SCROPE has charged him with a gross and absurd blunder in representing the crater in a direction whence it could not possibly be seen. Mr SCROPE supposes the basalts in the foreground to be those which he has himself figured in the ravine opposite Antraigues; whereas the view of FAUJAS is entirely confined to the little lateral valley leading to the Collet d'Aysac, as clearly appears from his ample, and generally accurate description.*

The lava of Aysac, after filling up in great part the basin of Antraigues, has followed the course of the Volane for several miles, as the numerous basaltic patches between that village and Vals sufficiently attest. These basalts are perhaps equalled by none in the Vivarais, or in any other part of Europe, as regards the exquisite perfection of the pillars of which they are composed; which, though they attain no great height, as at Jaujac and Thuez, are symmetrically polygonal, small in diameter, and beautifully jointed, affording beautiful cabinet specimens, particularly at the bridge across the Volane, two miles above Vals. This lava, like that of Burzet, appears to have been very fluid, and to have accumulated in the gorges to a greater height than it could afterwards retain. We find it, accordingly, applied in thin sheets upon the steep granite walls of the valley, forming, as usual, pillars perpendicular to the cooling surface, as in fig. 2 of Plate V.

Before finally quitting Antraigues, I must mention that the summit of the hill, a little to the ESE. of the village, is occupied by a formation of basalt (which is indicated in the map as a lava), but which appears rather to belong to the ancient basalts of the Coyrans and Mesilhac. It is composed chiefly of loose masses, excessively hard and heavy, containing olivine, chalcedony, and other substances, and decomposing into an earthy matter. It gives no indication of having flowed into the valley, and was probably formed before the valleys existed.

The village of Vals is pleasantly situated on the river Volane, below the last traces of the lava of Aysac, and not far from the junction of the gneiss with the secondary rocks. Two mineral springs issue from the gneiss at the river's side, having temperatures of 59°·5 and 61°. They are charged with carbonic acid, and contain a little iron and soda. They are frequented in summer by visitors even from a considerable distance, and, in consequence, the inn or hôtel at Vals is the only tolerable one in the whole district, and, as such, will be duly estimated by every traveller who has spent some time in the Vivarais. It is also a good centre for making excursions.

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* Recherches, p. 296–298.

One valley remains to be described, which I shall call the valley of La Bastide, from its principal village. It is intermediate between the valleys of Burzet and Antraigues, and may easily be reached from either of them; but by ascending from Vals (which is but a little way below where it unites with the valley of the Volant) it may be conveniently examined in its whole length. The first portion for several miles presents no trace of lava; but when we get opposite to the position of the Coupe d'Aysac, the valley opens and becomes extremely beautiful. A spring from the granite rising here, which discharges carbonic acid, has a temperature of 53°·2. A mile or two above this, we came at once upon an extensive lava deposit, excavated by the torrent, and commanding a pleasing view of the hamlet of La Bastide, with its water-mills, and the ruins of its chateau, which belonged to the Comte d'Antraigues, an extensive proprietor in this country, including nearly all the Bois de Cuze, who was expelled by the populace at the first French revolution, his house destroyed, and his lands divided.

The lava continues with but very little interruption for several miles, from La Bastide up the valley, indicating clearly a peculiar source of its own. I had previously visited it in 1839, and (more fortunate than in the case of the Burzet lava) I had fairly hunted it to cover, tracing it in an irregular stream up the face of the ridge of lofty hills separating the valleys of La Bastide and Antraigues, about three miles above the former village (Plate VI., fig. 2). The summit from which it is derived, I then learnt, was called the Pic de l'Etoile. My time did not allow me to ascend farther; but on a subsequent day I made an expedition expressly by the valley of Antraigues, and ascended the *right* bank of the Volane for four or five miles above that village, when I judged that I must be nearly opposite the volcano (of which, however, there is no trace visible from below on the east side). I commenced the ascent from the hamlet of La Viole (see the map), which proved to be excessively steep, and for a height of nearly 2000 feet, the latter part of the way being through tangled and nearly pathless brushwood, frequented only by a few charcoal burners. At less than one-third of the ascent I found, in a little ravine, a flow of lava, which proved that I had not mistaken my point, and from under it issued a fine spring, having a temperature of 45°, affording another proof (see p. 28) of the complete refrigeration of the lavas. This stream does not reach the bed of the Volane. As I followed it up, it constantly occupied the bed of the water-course, and soon led me amongst slag and cinders, which constitute the steep part of the ascent. At the top I found the summit of the chain occupied for some distance by volcanic products. The granite, however, resumes the higher position. On examining from the top the course of the stream of lava descending towards La Bastide, I perceived that it originated still farther beyond the granite; and, proceeding northwards, I welcomed the sight of a crater, of which hitherto I had perceived no trace. It is rather singularly situated, occupying the whole breadth of the ridge, in a sort of depression or *col*, at a point where the ridge makes, at the same time, a bend, as sketched in Plate V., fig. 1 (which was not drawn on the spot, but immediately after from memory).

Whatever may be thought of the crater of Pal near Montpezat, *this* one is undoubtedly blown out of granite, and entirely formed in it; for I do not think that the scoriæ constitute a considerable mass upon any side, and there is no outlet or imperfection upon any side except that to the westward, by which the lava has run into the valley of La Bastide. The crater is of a beautiful elliptical form, and lies exactly between two ravines on either side of the ridge; the imperfect lava stream towards the last no doubt flowed first. The elevation above the sea is 4204 feet, or somewhat higher than the crater of Pal. It commands a fine view, and here I first caught a glimpse of the probable origin of the lava of Burzet, which two years later I was enabled to confirm. This volcano, like that of La Fiollonge, appears to have been undiscovered by any previous geologist, and as an example of a crater in granite it is undoubtedly remarkable. I was forced to abridge my observations on account of a very violent thunderstorm, which covered every thing with mist, and compelled my retreat. The lava on the side of La Bastide accumulated to a considerable thickness, but it is not remarkably columnar. In some places it approaches almost to the character of obsidian. The character of the valley is cultivated and pleasing. Groups of well-built houses are studded over the slopes, amidst groves of chestnut, with neat and well-watered gardens adjoining. In all this country, as well as in the *Haute Loire*, masonry is, or has been, much attended to.

From the village of La Bastide, a pleasing route may be made into the lower part of the valley of Burzet, by crossing the Col of Juvinas, which commands an excellent view in both directions; and, in particular, of the Coupe d'Aysac. Juvinas is known as being near the spot where a large meteoric stone fell from the sky not many years ago, in broad daylight, and in the presence of several witnesses. I did not lose the opportunity of making every inquiry relative to so rare and interesting an occurrence, and conversing with those who had seen it. The exact spot is a hamlet called Le Creux de Libounez, between Juvinas and St Pierre Colombier. At the time of my inquiry (1839), every one spoke of it as a recent occurrence, and one never to be forgotten. The field was immediately shewn to me, a small enclosure just below the village. I inquired for the actual spectators of the fall. DOLMAAS, who had been mentioned to me, was dead; but, with some difficulty, I found two brothers named SERRE, who were working with some others in their potato field when the stone fell amongst them. One of these men gave me in his *patois* a most animated account of the scene, and of their terror. With the aid of an interpreter,* I extracted the following particulars, agreed upon

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* The native language of this country, as well as of the Haute Loire, is an almost unintelligible *patois*. It is more Italian or Latin than French, and is probably the remains of the old language of Provence. The following Italian phrases struck my ear, "un' ora e mezzo,"—"Aspett' un poc,"—"non ho mai stato,"—"Perchè."

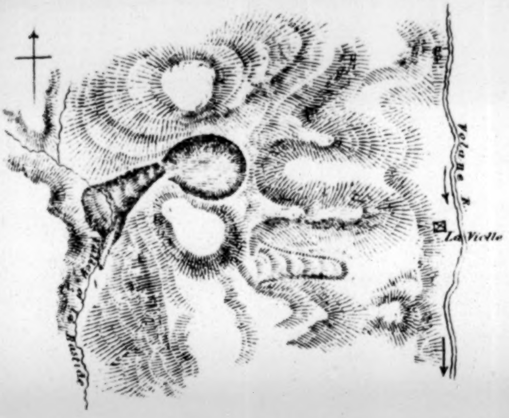


Fig 1 p 36 Pic de l'Etoile

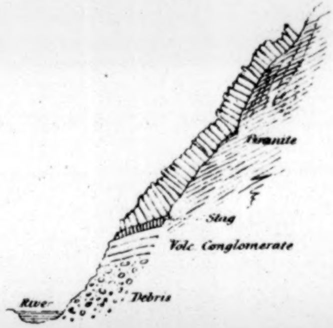


Fig 2 p 34 Section in the course of the Volane near Antraigues.

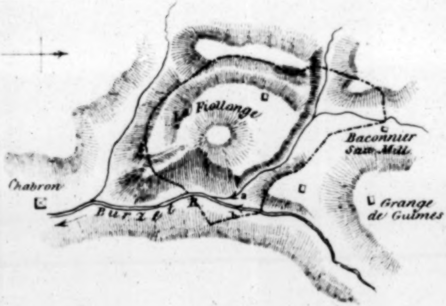


Fig 3 p 33 Crater of Fiollonge
a Cascade of Raipis b Small Cascade
.-.-.-.-. Limit of Volcanic formation

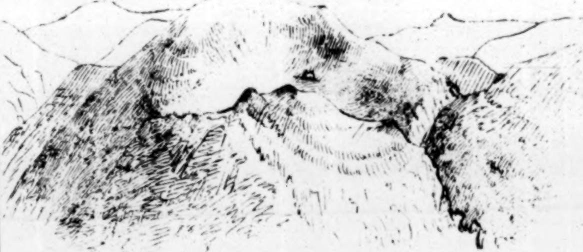
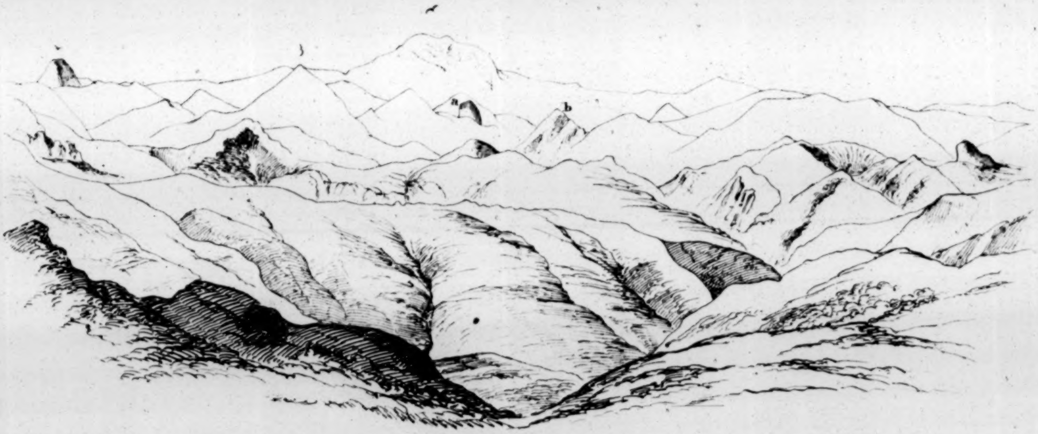


Fig 4 Crater of Fiollonge Elevation p. 33

Fig 5. p 6 and 33 View from Mesilhac



♈ Gerbier des Jonts. ʒ Charac ♂ Mezenc a Roche des Saulrons b La Pinede

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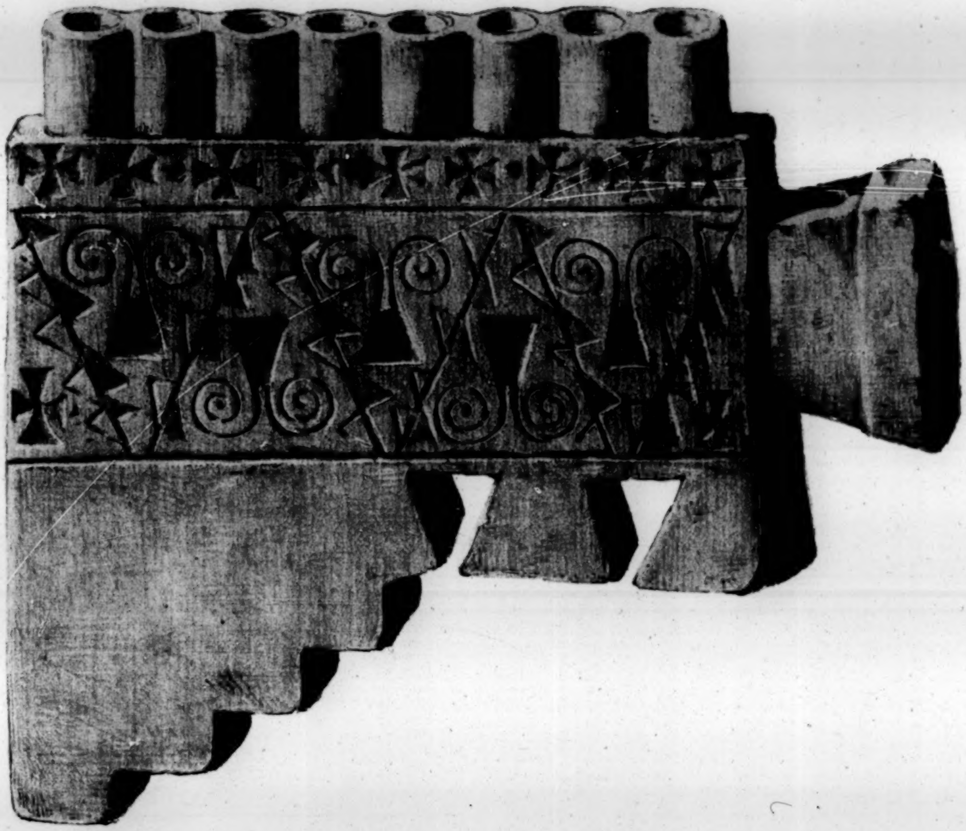

PERUVIAN SYRINX.

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by those present whom I questioned The stone fell on the 15th June 1821, whilst the sky was clear and the wind north; the hour was half-past four in the afternoon. A long rolling noise was heard, then an explosion like a cannon, which occurred five minutes before the stone fell. It touched the ground within a few feet of them, perforating it to a depth of 7 palms (about 5½ feet) *in a vertical direction*. It burnt the ground to a cindery state. No lightning accompanied the fall. The men were frightened, but not stunned; the noise was heard at a great distance; a man present said that he had heard it at Argentière (five French leagues' distant in a straight line). The people of Libounez thought it was *the devil which had fallen*, and did not venture to dig up the stone for seven days, when it was sprinkled with holy water by the priest. The hole was exactly the size of the stone; there was no scattering of earth. The course of the stone was from the NW.; (this is difficult to reconcile with its having penetrated *vertically*, perhaps the direction in which the noise was heard is meant.) The stone had wedged itself between two others, and could not be removed without breaking it. It weighed 220 pounds, as I was told by the man who had weighed it. It was sold for six francs; but the fragments have been so dispersed, that I with difficulty obtained one or two morsels, although I inquired for them in all the surrounding valleys.

Such was the circumstantial, and apparently authentic narrative, which I gathered from the spectators of this most curious occurrence; and they are entirely corroborated by a manuscript account (or *procès-verbal*), drawn up by the Maire of the Commune of Juvinas, and forwarded by him to the Prefecture of Privas, where I subsequently discovered and copied it. At Privas I also found, amongst a mass of mineral rubbish, preserved at the Prefecture, a small but characteristic specimen of the meteorite itself, which resembles the more ordinary varieties, and is coated with a superficial glossy black vitrification. The meteor of Juvinas is one of a very few which have fallen so near to intelligent spectators as actually to endanger their lives.

The following Table contains a summary of the altitudes referred to in the preceding paper. They were all obtained by myself by means of the barometer. Owing to the distance of the fundamental station, Marseilles, they have no pretension to more than a moderate approximation to the truth.

TABLE of HEIGHTS in the VIVARAIS *above the Level of the Mediterranean, barometrically determined in* 1839 *and* 1841 (*the latter are marked thus* *).

		Metres.	Eng. Feet.
*	Alignon and Ardêche, Surface of Lava at Junction of,	341	1117
	Antraigues, Bridge,	421	1383
	Aubenas, Hotel, second floor,	316	1035
*	... ... first floor,	311	1019
	Aysac, Coupe d', Summit of Crater,	805	2640
*	Bauzon Volcano, Summit,	1500	4922
*	Burzet, Village,	553	1815
	Chambon, Lava of, highest part,	1260	4134
*	Champ Raphael, Church,	1344	4409
*	Fiollonge, Crater,	1121	3678
*	Jaujac, Carbonic Acid Spring,	467	1531
	... Surface of Lava at Village of,	431	1413
*	... ...	413	1354
	... Top of Crater,	587	1923
	La Bastide, Village,	592	1942
	Libounez, Village,	523	1716
*	Mesilhac Church,	1155	3791
	Montpezat, Bridge of Fontaulier,	468	1534
	... Gravenne, highest part,	831	2727
	... ... lowest part,	800	2626
	... Village (6 obs.),	566	1857
*	Neyrac, Carbonic Acid Spring at Village of,	407	1359
	... Volcano of,	664	2178
	Pal, Crater of, Water Drainage,	1186	3893
	... ... highest part,	1383	4537
	Pic de l'Etoile,	1282	4204
	Pont de la Beaume,	327	1073
*	Thuez, Base of Lava Cliff at,	409	1343
	... Inn, second Floor,	464	1522
*	... ...	478	1569
	... Mean adopted,	471	1545
*	... Level of the Ardêche at Pont du Diable,	377	1237
*	... Surface of Lava at,	462	1514
*	... Volcano of Mouleyres Summit,	618	2026
*	Vals, Inn,	235	773

II.—*On a Process in the Differential Calculus, and its application to the Solution of certain Differential Equations.* By the Rev. P. KELLAND, *M.A., F.R.SS.L. & E., F.C.P.S., late Fellow of Queen's College, Cambridge; Professor of Mathematics, &c., in the University of Edinburgh.*

(Read 17th December 1849.)

The facilities which are afforded by the introduction of the function $\overline{|\;\;}$ into certain classes of Differential expressions are well known. This function has effected the combination and generalization of Problems, which, although found to be capable of solution in particular cases, were regarded rather as isolated and exceptional forms, than as integral parts of some comprehensive expression. But the subject is far from exhausted. Some of the most important Differential Equations have never as yet been solved by a general method. The present Memoir is intended to supply this defect. The process employed differs little from that which I have previously exhibited; but the range of Problems which it embraces is much more extensive, and the Problems themselves are of a more important character.

SECTION I. PRELIMINARY THEOREMS.

1. Let $$y=e^{-a x^r},$$

then $$\left(\frac{1}{x^{r-1}}\frac{d}{d x}\right) y=-a r y,$$

from which it follows, since the operation reproduces the function itself, that $\left(\frac{1}{x^{r-1}}\frac{d}{d x}\right)^{\mu} y=(-a r)^{\mu} y$ (1.), whatever μ may be.

It is necessary to observe, that if μ be negative, the above equation takes no cognizance of functions of integration, which would be introduced by means of the added arbitrary constants; and this remark applies to all our processes.

In the equation $\int_0^\infty e^{-\theta}\,\theta^{n-1}\,d\theta=\overline{|n},$

let $$\theta=a x^r,$$

then $$\int_0^\infty e^{-a x^r} a^{n-1} x^{r(n-1)} x^r \, d a=\overline{|n}$$

or $$\frac{\overline{|n}}{x^{r n}}=\int_0^\infty a^{n-1} e^{-a x^r} d a,$$

or, which is the same thing,

$\dfrac{\left|\overline{\tfrac{m}{r}}\right.}{x^m}=\int_o^\infty a^{\frac{m}{r}-1}e^{-a x^r}\,d a$; from which it follows, by equation (1.), that

$$\left|\overline{\tfrac{m}{r}}\right.\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^\mu\cdot\frac{1}{x^m}=\int_o^\infty a^{\frac{m}{r}-1}(-a r)^\mu e^{-a x^r}\,d a$$

$$=(-r)^\mu\frac{1}{x^{m+r\mu}}\int\theta^{\frac{m+r\mu}{r}-1}e^{-\theta}\,d\theta$$

$$=(-r)^\mu\frac{\left|\overline{\tfrac{m+r\mu}{r}}\right.}{x^{m+r\mu}}$$

Consequently, $$\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^\mu\frac{1}{x^m}=(-r)^\mu\frac{\left|\overline{\tfrac{m+r\mu}{r}}\right.}{\left|\overline{\tfrac{m}{r}}\right.}\,\frac{1}{x^{m+r\mu}}\ \ldots\ (2.)$$

Let u be a function, supposed to be capable of being represented under the form of a series of functions, such as $e^{-a x^r}$; then we shall have

$$x^{\mu r}\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^\mu u=(-r)^\mu\frac{\left|\overline{-\tfrac{D}{r}+\mu}\right.}{\left|\overline{-\tfrac{D}{r}}\right.}\,u\ \ldots\ldots\ (3.)$$

2. Since $$\frac{1}{x^{r-1}}\frac{d}{dx}u=\frac{d u}{x^{r-1}d x}=\frac{r\,d u}{d x^r}$$

it follows that $$\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^\mu u=r^\mu\left(\frac{d}{dx^r}\right)^\mu u$$

$$\therefore\qquad x^{\mu r}\left(\frac{d}{d\,.\,x^r}\right)^\mu u=(-1)^\mu\frac{\left|\overline{-\tfrac{D}{r}+\mu}\right.}{\left|\overline{-\tfrac{D}{r}}\right.}\,u\ \ldots\ldots\ (4.)$$

3. If we write $d_r u$ for $\dfrac{1}{x^{r-1}}\dfrac{d}{dx}u$; then from the last Art. it will be evident, without demonstration, that

$$d_r^\mu(u v)=v\,d_r^\mu u+\mu\,d_r v\,d_r^{\mu-1}u+\frac{\mu(\mu-1)}{1\,.\,2}d^2{}_r v\,d_r^{\mu-2}u+\&c.$$

4. Let $\phi(d_r)$ be any function of d_r capable of expansion in the form $\phi(d_r)=\Sigma\,a\,d_r^\mu$; then, by the last Article

$$\phi(d_r)(u v)=\Sigma\,a\,\{v\,a_r^\mu u+\mu\,d_r v\,d_r^{\mu-1}u+\&c.\}$$

$$-v\,(\Sigma\,a\,d_r^\mu)\,u+d_r v\,(\Sigma\,a\,\mu\,d_r^{\mu-1})\,u+\&c.$$

$$=v\,\phi(d_r)\,u+d_r v\,\phi'(d_r)\,u+\frac{1}{1\,.\,2}\,d^2{}_r v\,\phi''(d_r)\,u+\&c.$$

where $\phi'(d_r)$ is the differential coefficient of $\phi(d_r)$ with respect to d_r.

5. In Art. 3, let $v=e^{\beta x^r}$, then

$$d_r^{\mu}\,(e^{\beta x^r}u)=e^{\beta x^r}\{d_r^{\mu}u+\mu\beta r\,d_r^{\mu-1}u$$
$$+\frac{\mu(\mu-1)}{1\,.\,2}(\beta r)^2\,d_r^{\mu-2}u+\&c.\}$$
$$=e^{\beta x^r}\,(d_r+\beta r)^{\mu}u\,;$$

and hence, generally, as in Art. 1.,

$$\phi(d_r)\{e^{\beta x^r}u\}=e^{\beta x^r}\,\phi(d_r+\beta r)\,u\;.\;.\;.\;.\;.\;(5.)$$

6. Let $\phi(x)=\Sigma\,\mathrm{B}\,e^{\beta x^r}$, then

$$\phi(x)\,\psi(d_r)u=\Sigma\,\mathrm{B}\,e^{\beta x^r}\,\psi(d_r)\,u$$
$$=\Sigma\,\mathrm{B}\,\psi(d_r-\beta r)\,e^{\beta x^r}u\ \text{ by (5)}$$
$$=\Sigma\,\mathrm{B}\{\psi(d_r)-\psi'(d_r)\,\beta r+\psi''(d_r)\frac{(\beta r)^2}{1\,.\,2}-\&c.\}\,e^{\beta x^r}u$$
$$=\psi(d_r)\{\Sigma\mathrm{B}\,e^{\beta x^r}u\}-\psi'(d_r)\{d_r(\Sigma\,\mathrm{B}e^{\beta x^r})u\}+\&c.$$
$$=\psi(d_r)\{\phi(x)\,u\}-\psi'(d_r)\{d_r\,\phi(x)\,.\,u\}$$
$$+\frac{1}{1\,.\,2}\psi''(d_r)\{d_r^2\,\phi(x)\,.\,u\}-\ \&c.$$

7. It is easily seen that, if m be a whole number,

$$d_r^{-m}\log x=\frac{r^{-m}x^{rm}}{|\overline{1+m}}\left\{\log x-\frac{1}{r}\left(\frac{1}{1}+\frac{1}{2}+\&c+\frac{1}{m}\right)\right\};$$

and that in other cases,

$$d_r^{\mu}\log x=(-r)^{\mu-1}\,|\overline{\mu}\,x^{-r\mu}$$

8. The operation denoted by d_r^{-m} must not be confounded with $\left(x^{r-1}\int dx\right)^m$: it is, in reality, $\left(\int x^{r-1}dx\right)^m$. The one expression can be deduced from the other, thus:—

$$\left(x^{r-1}\int dx\right)^m\log x=x^{r-1}d_r^{-m}x^{-(r-1)}\log x=x^{r-1}d_r^{-m}\frac{1}{p}\left(x^{p-r+1}-x^{q-r+1}\right)$$
$$=\frac{(-r)^{-m}}{p}\left\{\frac{|\overline{-\left(\frac{p-r+1}{r}+m\right)}}{|\overline{-\frac{p-r+1}{r}}}\,x^p-\frac{|\overline{-\left(\frac{q-r+1}{r}+m\right)}}{|\overline{-\frac{q-r+1}{r}}}\,x^q\right\}x^{rm}$$

where p, q, and $\frac{q}{p}$ are indefinitely small.

Hence $$\left(x^{r-1}\int dx\right)^m \log x = \frac{x^{rm}}{p\,r^m}\left\{\frac{|\overline{\tfrac{1+p}{r}}}{|\overline{\tfrac{1+p}{r}+m}}\,x^p - \frac{|\overline{\tfrac{1+q}{r}}}{|\overline{\tfrac{1+q}{r}+m}}\,x^q\right\}$$

$$= x^{rm}r^{-m}\frac{|\overline{\tfrac{1}{r}}}{|\overline{\tfrac{1}{r}+m}}\left\{\log x - \left(\frac{1}{1}+\frac{1}{1+r}+\frac{1}{1+2r}+\text{\&c.}+\frac{1}{1+(m-1)r}\right)\right\}$$

9. *To find the differential equation, on the solution of which depends the value of* $d^{\mu}_{r}\,e^{ax}$.

Since $e^{ax}=1+ax+\frac{a^2x^2}{1\,.\,2}+\text{\&c.}$

$$d^{\mu}_{r}\,e^{ax}=(-r)^{\mu}x^{-r\mu}\left\{ax\frac{|\overline{\tfrac{-1+r\mu}{r}}}{|\overline{-\tfrac{1}{r}}}+\frac{a^2x^2}{1\,.\,2}\frac{|\overline{\tfrac{-2+r\mu}{r}}}{|\overline{-\tfrac{2}{r}}}+\text{\&c.}\right\}$$

This equation can be made to depend on the solution of a differential equation, when r is a whole number; for, in that case, the terms will recur after the rth. To reduce the equation, we observe that

$$\frac{|\overline{\tfrac{-(r+1)+r\mu}{r}}}{|\overline{-\tfrac{r+1}{r}}}=\frac{r+1}{r+1-r\mu}\,\frac{|\overline{\tfrac{-1+r\mu}{r}}}{|\overline{-\tfrac{1}{r}}}\text{, and so on.}$$

Hence $$d^{\mu}_{r}\,e^{ax}=(-r)^{\mu}x^{-r\mu}\left[\frac{|\overline{\tfrac{-1+r\mu}{r}}}{|\overline{-\tfrac{1}{r}}}\left(ax+\frac{r+1}{r+1-r\mu}\,\frac{a^{r+1}x^{r+1}}{|\underline{r+1}}+\text{\&c.}\right)\right.$$

$$\left.+\frac{|\overline{\tfrac{-2+r\mu}{r}}}{|\overline{-\tfrac{2}{r}}}\left(\frac{a^2x^2}{1\,.\,2}+\frac{r+2}{r+2-r\mu}\,\frac{a^{r+2}x^{r+2}}{|\underline{r+2}}+\text{\&c.}\right)+\text{\&c., \&c.,}\right]\ldots\ldots(A)$$

Let $$y=x^{1-r\mu}+\frac{a^rx^{r+1-r\mu}}{|\underline{r+1}}\,\frac{r+1}{r+1-r\mu}$$

$$+\frac{a^{2r}x^{2r+1-r\mu}}{|\underline{2r+1}}\,\frac{(r+1)\,(2r+1)}{(r+1-r\mu)\,(2r+1-r\mu)}+\text{\&c.}\ldots\ldots\ldots;\text{ then}$$

$$\frac{d^{r-1}}{dx^{r-1}}\left(x^{r\mu}\frac{dy}{dx}\right)=a^r\left\{\frac{x}{1}+\frac{a^rx^{r+1}(r+1)}{|\underline{r+1}(r+1-r\mu)}+\text{\&c.}\right\}$$

$$=a^rx^{r\mu}y$$

Again, if $$y=\frac{1}{1\,.\,2}x^{2-r\mu}\frac{|\overline{\tfrac{-2+r\mu}{r}}}{|\overline{-\tfrac{2}{r}}}+\frac{a^rx^{r+2-r\mu}}{|\underline{r+2}}\,\frac{|\overline{\tfrac{-(r+2)+r\mu}{r}}}{|\overline{-\tfrac{r+2}{r}}}+\text{\&c.}$$

$$=\frac{\overline{|\,\frac{-2+r\mu}{r}}}{\overline{|\,-\frac{2}{r}}}\left\{\frac{1}{1\,.\,2}x^{2-r\mu}+\frac{r+2}{r+2-r\mu}\,\frac{a^r x^{r+2-r\mu}}{\underline{|r+2}}\right.$$

$$\left.+\frac{a^{2r}x^{2r+2-r\mu}}{\underline{|2r+2}}\,\frac{(r+2)(2r+2)}{(r+2-r\mu)(2r+2-r\mu)}+\text{\&c.}\right\}$$

$$\therefore\quad\frac{d^{r-1}}{dx^{r-1}}\left(x^{r\mu}\frac{dy}{dx}\right)=\frac{\overline{|\,\frac{-2+r\mu}{r}}}{\overline{|\,-\frac{2}{r}}}\,a^r\left\{\frac{x^2}{1\,.\,2}+\frac{a^r x^{r+2}}{\underline{|r+2}}\,\frac{r+2}{r+2-r\mu}+\text{\&c.}\right\}$$

$$=a^r x^{r\mu}y\;.\;.\;.\;.\;(\text{B}).$$

The same process will give the same result for all the terms of this expansion except the last. But as $\frac{\overline{|\,-1+\mu}}{\overline{|\,-1}}$ is a factor of that term, it is evidently 0.

Consequently, the equation $\frac{d^{r-1}}{dx^{r-1}}\left(x^{r\mu}\frac{dy}{dx}\right)=a^r x^{r\mu}y$ gives the complete value of $y=d_r^{\mu}\,e^{ax}$, it being observed that one of the constants of integration is 0. The other constants must be determined by means of equation (A).

10. To find $d_r^{\mu}\cos ax$.

Since $$\left.\cos ax=1-\frac{a^2}{\underline{|2}}x^2+\frac{a^4}{\underline{|4}}x^4-\text{\&c.}\right\}$$

if $-a^2=b$, we shall have

$$\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^{\mu}\cos ax=(-r)^{\mu}\left\{\frac{b}{\underline{|2}}\,\frac{\overline{|\,\frac{-2+r\mu}{r}}}{\overline{|\,-\frac{2}{r}}}\,x^{2-r\mu}\right.$$

$$\left.+\frac{b^2}{\underline{|4}}\,\frac{\overline{|\,\frac{-4+r\mu}{r}}}{\overline{|\,-\frac{4}{r}}}\,x^{4-r\mu}+\frac{b^3}{\underline{|6}}\,\frac{\overline{|\,\frac{-6+r\mu}{r}}}{\overline{|\,-\frac{6}{r}}}\,x^{6-r\mu}+\text{\&c.}\right\}$$

It remains to express this series in the form of a differential equation when r is a whole number.

Let s be any whole number less than $r+1$, $\text{S}=(-r)^{\mu}\,b^s\,\frac{\overline{|\,\frac{-2s+r\mu}{r}}}{\overline{|\,-\frac{2s}{r}}}$; then the series may be written

$$y=\text{S}\left\{\frac{x^{2s-r\mu}}{\underline{|2s}}+\frac{b^r}{\underline{|2r+2s}}\,\frac{(r+2s)(2r+2s)}{(r+2s-r\mu)(2r+2s-r\mu)}\,x^{2r+2s-r\mu}+\text{\&c.}\right\}$$

$$\therefore\;\frac{d^{r-1}}{dx^{r-1}}\left(x^{r\mu}\frac{dy}{dx}\right)=\text{S}\left\{\frac{2s-r\mu}{2s}\,\frac{x^{2s-r}}{\underline{|2s-r}}+\frac{b^r}{\underline{|2s+r-1}}\,\frac{x^{2s+r}}{r+2s-r\mu}\right.$$

$$\left. + \frac{b^{2r}}{\underline{|2s+3r-1}} \frac{(r+2s)(2r+2s)x^{2s+3r}}{(r+2s-r\mu)(2r+2s-r\mu)(3r+2s-r\mu)} + \&c. \right\}$$

$$\frac{d}{dx}x^{-r\mu}\frac{d^{r-1}}{dx^{r-1}}x^{r\mu}\frac{dy}{dx} = \mathrm{S}\left\{\frac{(2s-r\mu)(2s-r-r\mu)}{2s(2s-r)}\frac{x^{2s-r-1-r\mu}}{\underline{|2s-r-1}}\right.$$

$$\left. + \frac{b^r}{\underline{|2s+r-1}} x^{2s+r-1-r\mu} + \frac{b^{2r}}{\underline{|2s+3r-1}} \frac{(r+2s)(2r+2s)x^{2s+3r-1-r\mu}}{(r+2s-r\mu)(2r+2s-r\mu)} + \&c. \right\}$$

$$\frac{d^{r-1}}{dx^{r-1}}x^{r\mu}\frac{d}{dx}x^{-r\mu}\frac{d^{r-1}}{dx^{r-1}}x^{r\mu}\frac{dy}{dx} = \mathrm{S}\left\{\frac{(2s-r\mu)(2s-r-r\mu)}{2s(2s-r)\underline{|2s-2r}}x^{2s-2r}\right.$$

$$\left. + \frac{b^r}{\underline{|2s}}x^{2s} + \frac{b^{2r}}{\underline{|2r+2s}}\frac{(r+2s)(2r+2s)}{(r+2s-r\mu)(2r+2s-r\mu)}x^{2s+2r} + \&c. \right\}$$

$$= \mathrm{S}\frac{(2s-r\mu)(2s-r-r\mu)}{2s(2s-r)\underline{|2s-2r}}x^{2s-2r} + b^r x^{r\mu} y.$$

If $s \angle r, \quad \dfrac{1}{\underline{|2s-2r}} = 0$

If $s = r, \quad \mathrm{S}\dfrac{(2s-r\mu)(2s-r-r\mu)}{2s(2s-r)\underline{|2s-2r}}x^{2s-2r} = (-r)^\mu \dfrac{\overline{\left|-\frac{2s}{r}+\mu+2\right.}}{\overline{\left|-\frac{2s}{r}+2\right.}}$

$$= (-r)^\mu \frac{\overline{|\mu}}{\overline{|0}} = 0\,;$$

Therefore, generally, if $y = d_r^\mu \cos ax$, we have

$$\frac{d^{r-1}}{dx^{r-1}}x^{r\mu}\frac{d}{dx}x^{-r\mu}\frac{d^{r-1}}{dx^{r-1}}x^{r\mu}\frac{dy}{dx} = (-a^2)^r x^{r\mu} y\,;$$

or, if $x^{r\mu}y = z$, the equation may be written

$$\left(\frac{d^{r-1}}{dx^{r-1}}x^{r\mu}\frac{d}{dx}x^{-r\mu}\right)^2 z = (-a^2)^r z, \quad \ldots \text{(C)}$$

a form which exhibits its relation with the differential equation resulting from e^{ax}.

11. There exist numerous relations between the differential coefficients of a function; such as the following:—

Since $\quad x^{\mu r}\left(\dfrac{1}{x^{r-1}}\dfrac{d}{dx}\right)^\mu u = (-r)^\mu \dfrac{\overline{\left|-\frac{\mathrm{D}}{r}+\mu\right.}}{\overline{\left|-\frac{\mathrm{D}}{r}\right.}}u\,;$

let $\quad \phi\left(\dfrac{\mathrm{D}}{r}\right) = \dfrac{\overline{\left|-\frac{\mathrm{D}}{r}+\mu\right.}}{\overline{\left|-\frac{\mathrm{D}}{r}\right.}}$; then

$$\phi\left(\frac{\mathrm{D}}{r}+1\right)=\frac{\Gamma\left(-\frac{\mathrm{D}}{r}-1+\mu\right)}{\Gamma\left(-\frac{\mathrm{D}}{r}-1\right)}=\frac{\frac{\mathrm{D}}{r}+1}{\frac{\mathrm{D}}{r}-\mu+1}\,\phi\left(\frac{\mathrm{D}}{r}\right)$$

Hence
$$\phi\left(\frac{\mathrm{D}}{r}\right)=\frac{\Gamma\left(\frac{\mathrm{D}}{r}+1\right)}{\Gamma\left(\frac{\mathrm{D}}{r}-\mu+1\right)}$$

$$=x^{(\mu-1)r}\frac{\Gamma\left(\frac{\mathrm{D}}{r}+\mu\right)}{\Gamma\left(\frac{\mathrm{D}}{r}\right)}x^{-(\mu-1)r}$$

$$=x^{(\mu-1)r}\frac{\Gamma\left(\frac{-\mathrm{D}}{-r}+\mu\right)}{\Gamma\left(\frac{-\mathrm{D}}{-r}\right)}x^{-(\mu-1)r}$$

Hence
$$x^{\mu r}\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^{\mu}u=(-1)^{\mu}\,x^{-r}\left(x^{r+1}\frac{d}{dx}\right)^{\mu}x^{-(\mu-1)r}u,$$

a relation between differentials with positive and negative values of r.

Section II. Application of the preceding Theory to the Solution of Differential Equations.

12. The first example which I propose to give is the solution of equation (B), Art. 9.

Ex. 1. $\frac{d^{r-1}}{dx^{r-1}}\left(x^{r\mu}\frac{dy}{dx}\right)=a^r x^{r\mu}y$, where μ may be anything whatever.

Let
$$x^{r\mu}y=z,$$

$$\therefore\quad \frac{d^{r-1}}{dx^{r-1}}x^{r\mu}\frac{d}{dx}x^{-r\mu}z=a^r z$$

or
$$x^{r-1}\frac{d^{r-1}}{dx^{r-1}}x^{r\mu-1}\mathrm{D}\,x^{-r\mu}z=a^r x^{r-1}z$$

or
$$\mathrm{D}(\mathrm{D}-1)\;\ldots\;(\mathrm{D}-r+2)\times(\mathrm{D}-r\mu+1)\,e^{-\theta}z=a^r e^{r\theta}e^{-\theta}z$$

Let
$$\frac{z}{x}\text{ or }e^{-\theta}z=f\left(-\frac{\mathrm{D}}{r}\right)v\text{; then}$$

$$\mathrm{D}(\mathrm{D}-1)\;\ldots\;(\mathrm{D}-r+2)\times(\mathrm{D}-r\mu+1)f\left(-\frac{\mathrm{D}}{r}\right)v=$$

$$a^r e^{r\theta}f\left(-\frac{\mathrm{D}}{r}\right)v=a^r f\left(-\frac{\mathrm{D}}{r}+1\right)e^{r\theta}v.$$

Which equation is reduced to

$$D(D-1)\ .\ .\ .\ .\ (D-r+1)\,v=a^r e^{r\theta}\,v \qquad (1.)$$

by the condition $f\left(-\frac{D}{r}+1\right)=\frac{D-r\mu+1}{D-r+1}f\left(-\frac{D}{r}\right)$ (2.)

$$=\frac{-\frac{D}{r}-\frac{1}{r}+\mu}{-\frac{D}{r}-\frac{1}{r}+1}f\left(-\frac{D}{r}\right)$$

the solution of which equation is

$$f\left(-\frac{D}{r}\right)=\frac{\left|\overline{-\frac{D}{r}-\frac{1}{r}+\mu}\right.}{\left|\overline{-\frac{D}{r}-\frac{1}{r}+1}\right.}$$

whence $\frac{z}{x}$ or $x^{r\mu-1}y=\frac{\left|\overline{-\frac{D}{r}-\frac{1}{r}+\mu}\right.}{\left|\overline{-\frac{D}{r}-\frac{1}{r}+1}\right.}\,v$

$$=x^{r-1}\frac{\left|\overline{-\frac{D}{r}+\mu-1}\right.}{\left|\overline{-\frac{D}{r}}\right.}\,x^{-r+1}\,v$$

$$=(-r)^{-\mu+1}\,x^{\mu r-1}\,d_r^{\mu-1}\,x^{-r+1}\,v$$

Now v is the solution of equation (1); or, which is the same thing, of

$$\frac{d^r v}{d x^r}=a^r\,v$$

$\therefore$ $v=\mathrm{A}\,e^{ax}$ is a value of v; and hence, finally,

$$y=(-r)^{-\mu+1}\,\mathrm{A}\,d_r^{\mu-1}\frac{e^{ax}}{x^{r-1}}$$

$$=\mathrm{B}\,d_r^{\mu}\,e^{ax}\text{, where B is an arbitrary constant.}$$

It is evident that any other of the r roots of the equation $\frac{d^r v}{d x^r}=a^r v$ would satisfy the conditions, or that any one of the roots of a^r will produce the same result as a itself.

This example affords an excellent illustration of the process.

Let us take, in the next place, the equation which occurs in the theory of the Figure of the Earth; an equation which has been solved by Professor BOOLE, in his admirable Memoir in the *Philosophical Transactions* for 1844, p. 251, but whose process is necessarily restricted to integral values of i. The equation is—

Ex. 2. $$\frac{d^2 u}{d x^2}-q^2 u-\frac{i(i+1)\,u}{x^2}=0.$$

By the usual process, this equation becomes

$$\mathrm{D}(\mathrm{D}-1)u-q^2e^{2\theta}u-i(i+1)u=0$$

or

$$(\mathrm{D}+i)(\mathrm{D}-i-1)u-q^2e^{2\theta}u=0.$$

Let $u=e^{-i\theta}f\left(-\frac{\mathrm{D}}{2}\right)v$, then

$$(\mathrm{D}+i)(\mathrm{D}-i-1)e^{-i\theta}f\left(-\frac{\mathrm{D}}{2}\right)v-q^2e^{-i\theta}f\left(-\frac{\mathrm{D}}{2}+1\right)e^{2\theta}v=0$$

or

$$\mathrm{D}(\mathrm{D}-2i-1)f\left(-\frac{\mathrm{D}}{2}\right)v-q^2f\left(-\frac{\mathrm{D}}{2}+1\right)e^{2\theta}v=0$$

an equation which, by omitting the function of 0, is reduced to

$$\mathrm{D}(\mathrm{D}-1)v-q^2e^{2\theta}v=0 \qquad (1.)$$

by the condition

$$f\left(-\frac{\mathrm{D}}{2}+1\right)=\frac{\mathrm{D}-2i-1}{\mathrm{D}-1}f\left(-\frac{\mathrm{D}}{2}\right) \qquad (2.)$$

$$=\frac{-\frac{\mathrm{D}}{2}+i+\frac{1}{2}}{-\frac{\mathrm{D}}{2}+\frac{1}{2}}f\left(-\frac{\mathrm{D}}{2}\right)$$

$$\therefore \quad f\left(-\frac{\mathrm{D}}{2}\right)=\frac{\Gamma\left(-\frac{\mathrm{D}}{2}+i+\frac{1}{2}\right)}{\Gamma\left(-\frac{\mathrm{D}}{2}+\frac{1}{2}\right)}$$

and

$$u=e^{-i\theta}\frac{\Gamma\left(-\frac{\mathrm{D}}{2}+i+\frac{1}{2}\right)}{\Gamma\left(-\frac{\mathrm{D}}{2}+\frac{1}{2}\right)}v$$

$$=e^{-(i-1)\theta}\frac{\Gamma\left(-\frac{\mathrm{D}}{2}+i\right)}{\Gamma\left(-\frac{\mathrm{D}}{2}\right)}e^{-\theta}v$$

$$=(-2)^{-i}x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i\frac{v}{x}$$

But the solution of Equation (1.) is

$$v=\mathrm{A}e^{qx}+\mathrm{B}e^{-qx}:$$

Hence the complete solution of the given equation is

$$u=(-2)^{-i}x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i\frac{1}{x}(\mathrm{A}e^{qx}+\mathrm{B}e^{-qx})$$

This solution is susceptible of another form by Art. 11; for by that Article,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i\frac{v}{x}=(-1)^ix^{-(2i+2)}\left(x^3\frac{d}{dx}\right)^i\frac{v}{x^{2i-1}}$$

$$\therefore \quad u=2^{-i}\frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^i\frac{\mathrm{A}e^{qx}+\mathrm{B}e^{-qx}}{x^{2i-1}},$$

in which form the solution has been given by Professor BOOLE, when i is a whole number.

Ex. 3. *Solution of the Equation for Laplace's Functions.*

This equation has been reduced by Professor BOOLE, in the *Cambridge Mathematical Journal*, New Series, vol. i., p. 15, to the form

$$(1-\mu^2)\frac{d^2 v}{d\mu^2}+2(a-1)\mu\frac{dv}{d\mu}+\{n(n+1)-a(a-1)\}v=0 \ldots (1)$$

where u, the quantity sought, is connected with v by the equation

$$u=(1-\mu^2)^{-\frac{a}{2}}v \quad (2).$$

Putting $\mu=e^{\theta}$ and writing D for $\frac{d}{d\theta}$, we have

$$\{(e^{-2\theta}-1)\mathrm{D}(\mathrm{D}-1)+2(a-1)\mathrm{D}+n(n+1)-a(a-1)\}v=0$$

or
$$e^{-2\theta}\mathrm{D}(\mathrm{D}-1)v-(\mathrm{D}-\overline{a+n})(\mathrm{D}-\overline{a-n-1})v=0$$

or
$$\mathrm{D}(\mathrm{D}-1)v-(\mathrm{D}-a-n-2)(\mathrm{D}-a+n-1)e^{2\theta}v=0$$

Let, therefore, $v=f\left(-\frac{\mathrm{D}}{2}\right)w$ (3), and this equation becomes

$$\mathrm{D}(\mathrm{D}-1)f\left(-\frac{\mathrm{D}}{2}\right)w-(\mathrm{D}-a-n-2)(\mathrm{D}-a+n-1)f\left(-\frac{\mathrm{D}}{2}+1\right)e^{2\theta}w=0$$

or $\quad \mathrm{D}(\mathrm{D}-1)w-(\mathrm{D}-a-n-1)(\mathrm{D}-a-n-2)e^{2\theta}w=0$ (4), provided

$$f\left(-\frac{\mathrm{D}}{2}+1\right)=\frac{\mathrm{D}-a-n-1}{\mathrm{D}-a+n-1}f\left(-\frac{\mathrm{D}}{2}\right)$$

$$=\frac{-\frac{\mathrm{D}}{2}+\frac{a}{2}+\frac{n}{2}+\frac{1}{2}}{-\frac{\mathrm{D}}{2}+\frac{a}{2}-\frac{n}{2}+\frac{1}{2}}f\left(-\frac{\mathrm{D}}{2}\right)$$

an equation of which the solution is evidently

$$f\left(-\frac{\mathrm{D}}{2}\right)=\frac{\overline{\left|-\frac{\mathrm{D}}{2}+\frac{a}{2}+\frac{n}{2}+\frac{1}{2}\right.}}{\left|-\frac{\mathrm{D}}{2}+\frac{a}{2}-\frac{n}{2}+\frac{1}{2}\right.} \qquad (5.)$$

and by (3)
$$v=f\left(-\frac{\mathrm{D}}{2}\right)w$$

$$=\frac{\overline{\left|-\frac{\mathrm{D}}{2}+\frac{a}{2}+\frac{n}{2}+\frac{1}{2}\right.}}{\left|-\frac{\mathrm{D}}{2}+\frac{a}{2}-\frac{n}{2}+\frac{1}{2}\right.}w$$

$$=e^{-(n-a-1)\theta}\frac{\overline{\left|-\frac{\mathrm{D}}{2}+n\right.}}{\left|-\frac{\mathrm{D}}{2}\right.}e^{(n-a-1)\theta}w$$

$$= (-2)^{-n}\mu^{n+a+1}\left(\frac{1}{\mu}\frac{d}{d\mu}\right)^{n}\mu^{n-a-1}w \quad (6)$$

a result which is true, whether n be integral or fractional.

The value of w from Equation (4.) is easily found, and is given by Mr **Boole** in the form

$$w=(1+\mu)^{n+a}\psi(\phi)+(1-\mu)^{n+a}\chi(\phi)$$

whence that of u is found.

Let us take, as our next example, the equation which has been discussed by M. **Poisson** in the *Journal de l'Ecole Polytechnique*, cah. 17, p. 614; and by Professor **Boole** in the *Philosophical Transactions* for 1844, p. 254.

Ex. 4. $$(1-x^2)\frac{d^2u}{dx^2}+\left\{\frac{p+1}{x}-(4n-p+1)x\right\}\frac{du}{dx}-2n(2n-p)u=0. \quad . \; . \; . \; (1.)$$

The symbolical form of this equation is

$$u-\frac{(D+2n-2)(D+2n-2-p)}{D(D+p)}e^{2\theta}u=0. \quad . \; . \; . \; (2.)$$

Let $u=f\left(-\frac{D}{2}\right)v \quad . \; . \; . \; . \; (3.)$; then

$$f\left(-\frac{D}{2}\right)v-\frac{(D+2n-2)(D+2n-2-p)}{D(D+p)}f\left(-\frac{D}{2}+1\right)e^{2\theta}v=0 \quad . \; . \; (4.)$$

This last equation may be reduced to an integrable form in various ways:

1. By making $f\left(-\frac{D}{2}+1\right)=\frac{D+p}{D-1}\cdot\frac{D+2n-3}{D+2n-2-p}f\left(-\frac{D}{2}\right)$

or $$f\left(-\frac{D}{2}\right)=\frac{\lceil\overline{-\frac{D}{2}-\frac{p}{2}}}{\lceil\overline{-\frac{D}{2}+\frac{1}{2}}}\cdot\frac{\lceil\overline{-\frac{D}{2}-n+\frac{3}{2}}}{\lceil\overline{-\frac{D}{2}-n+1+\frac{p}{2}}}$$

$$=e^{\theta}\frac{\lceil\overline{-\frac{D}{2}-\frac{p}{2}-\frac{1}{2}}}{\lceil\overline{-\frac{D}{2}}}e^{-(2n-p-1)\theta}\frac{\lceil\overline{-\frac{D}{2}-\frac{p}{2}+\frac{1}{2}}}{\lceil\overline{-\frac{D}{2}}}e^{(2n-p-2)\theta}$$

or $$u=(-2)^{p}x^{-p}\left(\frac{1}{x}\frac{d}{dx}\right)^{-\frac{p+1}{2}}x^{-2n+2}\left(\frac{1}{x}\frac{d}{dx}\right)^{-\frac{p-1}{2}}x^{2n-p-2}v$$

and equation (4.) becomes

$$v-\frac{(D+2n-2)(D+2n-3)}{D(D-1)}e^{2\theta}v=0 \quad . \; . \; . \; (5.)$$

which is a known form, and thus the given equation is completely solved.

It is evident that our solution reduces the operation to that of ordinary differentiation or integration, when p is an odd integer, positive or negative. By varying the process, however, we can obtain other forms of the solution of this equation, which are reduced to ordinary differentiation or integration in other cases. The other forms are as follow:

2. By making $$f\left(-\frac{D}{2}+1\right)=\frac{D}{D+2n-2}f\left(-\frac{D}{2}\right)$$

or $$f\left(-\frac{D}{2}\right)=\frac{\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{D}{2}-n+1\right)}$$

$$=e^{-(2n-2)\theta}\frac{\Gamma\left(-\frac{D}{2}+n-1\right)}{\Gamma\left(-\frac{D}{2}\right)}e^{(2n-2)\theta}$$

and, consequently, $u=(-2)^{-(n-1)}\left(\frac{1}{x}\frac{d}{dx}\right)^{n-1}x^{2n-2}v$; where v is found from the equation of the first order $v-\frac{D+2n-2-p}{D+p}e^{2\theta}v=\left(f\left(-\frac{D}{2}\right)\right)^{-1}0.$

This solution is of the ordinary form, whenever n is a positive or negative integer.

By making $$f\left(-\frac{D}{2}+1\right)=\frac{D}{D+2n-2-p}f\left(-\frac{D}{2}\right)$$

or $$f\left(-\frac{D}{2}\right)=\frac{\Gamma\left(-\frac{D}{2}\right)}{\Gamma\left(-\frac{D}{2}-n+1+\frac{p}{2}\right)}$$

$$=e^{-(2n-2-p)}\frac{\Gamma\left(-\frac{D}{2}+n-1-\frac{p}{2}\right)}{\Gamma\left(-\frac{D}{2}\right)}e^{2n-2-p}$$

and $$u=(-2)^{-(n-\frac{p}{2}-1)}\left(\frac{1}{x}\frac{d}{dx}\right)^{n-\frac{p}{2}-1}x^{2n-p-2}v;$$

where v is found from the solution of the equation of the first order,

$$v-\frac{D+2n-2}{D+p}e^{2\theta}v=f\left(-\frac{D}{2}\right)^{-1}.0.$$

This solution is of the ordinary form when $n-\frac{p}{2}$ is an integer.

4. By making $$f\left(-\frac{D}{2}+1\right)=\frac{D+p}{D+2n-p-2}f\left(-\frac{D}{2}\right),$$

or $$f\left(-\frac{D}{2}\right)=\frac{\Gamma\left(-\frac{D}{2}-\frac{p}{2}\right)}{\Gamma\left(-\frac{D}{2}-n+\frac{p}{2}+1\right)}$$

$$=e^{-(2n-p-2)\theta}\frac{\Gamma\left(-\frac{D}{2}+n-p-1\right)}{\Gamma\left(-\frac{D}{2}\right)}e^{(2n-p-2)\theta}$$

and $$u=(-2)^{-(n-p-1)}x^{-p}\left(\frac{1}{x}\frac{d}{dx}\right)^{n-p-1}x^{2n-p-2}v,$$

where v is found from the solution of the equation of the first order,

$$v-\frac{D+2n-2}{D}e^{2\theta}v=f\left(-\frac{D}{2}\right)^{-1}0.$$

This solution is of the ordinary form when $n-p$ is an integer.

This last form appears to have been overlooked by Professor BOOLE.

It may be remarked, that we have omitted $f\left(-\frac{D}{2}\right)^{-1}0$ in the first solution. The reason is, that equation (5) being of the second order, will contain two arbitrary constants, and will thus render the solution of the given equation complete, without the introduction of any other terms.

Ex. 5. $$(1-x^2)\frac{d^2u}{dx^2}-(2m+1)x\frac{du}{dx}-(m^2-q^2)u=0.$$

This equation has been solved by Professor BOOLE; but it is requisite to shew that his solution is not confined to integral values of m.

The symbolical form of the equation is

$$u-\frac{(D+m-2)^2-q^2}{D(D-1)}e^{2\theta}u=0.$$

Let $$u=e^{-m\theta}f(-D)v;\text{ then}$$

$$f(-D)v-\frac{D-2)^2-q^2}{(D-m)(D-m-1)}f(-D+2)e^{2\theta}v=0.$$

By making $$f(-D)=\frac{D(D-1)}{(D-m)(D-m-1)}f(-D+2);$$

or $$f(-D)=\frac{\Gamma(-D+m)}{\Gamma(-D)}$$

this equation is reduced to

$$v-\frac{(D-2)^2-q^2}{D(D-1)}e^{2\theta}v=0\ (1)$$

and $$u=e^{-m\theta}\frac{\Gamma(-D+m)}{\Gamma(-D)}v=(-1)^{-m}\left(\frac{d}{dx}\right)^m v$$

$$=\left(\frac{d}{dx}\right)^m\left\{A\cos(q\sin^{-1}x)+B\sin(q\sin^{-1}x)\right\}$$

This equation is, however, susceptible of a solution in a different form from that which we have just exhibited.

Let $$u=f\left(-\frac{D}{2}\right)v,\quad\text{then}$$

$$f\left(-\frac{D}{2}\right)v-\frac{(D+m-2-q)(D+m-2+q)}{D(D-1)}f\left(-\frac{D}{2}+1\right)e^{2\theta}v=0$$

By making $f(-\frac{D}{2}+1)=\frac{(D-1)}{D+m-2+q}f\left(-\frac{D}{2}\right)$ . . . (1.)

this equation is reduced to the equation of the first order.

$$v-\frac{D+m-2-q}{D}e^{2\theta}v=f\left(-\frac{D}{2}\right)^{-1}0 \quad . . . (2.)$$

Also $f\left(-\frac{D}{2}\right)=\frac{\lceil\overline{-\frac{D}{2}+\frac{1}{2}}}{\lceil\overline{-\frac{D}{2}-\frac{m}{2}+1-\frac{q}{2}}}$

$$=e^{-(m-2+q)\theta}\frac{\lceil\overline{-\frac{D}{2}+\frac{m}{2}-\frac{1}{2}+\frac{q}{2}}}{\lceil\overline{-\frac{D}{2}}}e^{(m-2+q)\theta}$$

$$u=(-2)^{-(m+q-1)}x\left(\frac{1}{x}\frac{d}{dx}\right)^{m+q-1}x^{m+q-2}v$$

which is of the ordinary form when $m+q$ is a whole number.

The same is true if $-q$ be substituted for q.

SECTION III. COMPARISON OF PROCESSES.

We shall occupy this Section in the comparison of the different methods which may be employed in the solution of a given Differential Equation.

13. *To find the value of* $\lceil\overline{-D}\,.\,v$.

Suppose v expressed in the form

$$v=\frac{A}{x^n}+\frac{B}{x^{n+r}}+\&c.$$

$$=Ae^{-n\theta}+Be^{-(n+r)\theta}+\&c.$$

then $\lceil\overline{-D}\,.\,v=A\lceil\overline{n}\,e^{-n\theta}+B\lceil\overline{n+r}\,e^{-(n+r)\theta}+\&c.$

$$=A\frac{\lceil\overline{n}}{x^n}+B\frac{\lceil\overline{n+r}}{x^{n+r}}+\&c.$$

$$=\int_0^\infty e^{-ax}\,d\,a\,(A\,a^{n-1}+B\,a^{n+r-1}+\ldots)$$

$$=\int_0^\infty e^{-ax}\frac{d\,a}{a}\,(A\,a^{n}+B\,a^{n+r}+\&c.$$

Hence, if $v=f(x)$

$$\lceil\overline{-D}f(x)=\int_0^\infty e^{-ax}\frac{d\,a}{a}\,f\left(\frac{1}{a}\right)$$

14. All equations of the form $y - m x^r \frac{d^n y}{d x^n} = X$ can be converted into equations in $\frac{d^r v}{d x^r}$.

For $$y - m x^{r-n} (-1)^n \frac{\overline{|-D+n}}{\overline{|-D}} y = X$$

Let $$y = \overline{|-D} \,.\, v, \quad \text{then}$$

$$\overline{|-D} \,.\, v - m x^{r-n} (-1)^n \,\overline{|-D+n}\, v = X$$

or $$v - m(-1)^n \frac{\overline{|-D+r}}{\overline{|-D}} e^{(r-n)\theta} v = (\overline{|-D})^{-1} X$$

or $$v - m(-1)^{-(r-n)} x^r \left(\frac{d}{dx}\right)^r x^{r-n} v = (\overline{|-D})^{-1} X$$

or, if $$x^{r-n} v = z,$$

$$z - m(-1)^{-(r-n)} x^{2r-n} \frac{d^r z}{d x^r} = x^{r-n} (\overline{|-D})^{-1} X$$

wherefore the above equation is reduced to an ordinary differential equation whatever be x, provided r is an integer.

As an example, let us take the simplest case, of which the solution will be found at pp. 257, 258 of my Memoir on General Differentiation, Part III.

Ex. 1. $$y - a\sqrt{-1}\, x \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0$$

Suppose $$y = \overline{|-D}\, v,$$

then $$v\sqrt{x} + a\sqrt{x} \frac{\overline{|-D+1}}{\overline{|-D}} (v\sqrt{x}) = C\sqrt{x}$$

or $$z - a x^{\frac{3}{2}} \frac{dz}{dx} = C\sqrt{x}, \text{ where } z = v\sqrt{x};$$

$$\therefore \quad z = e^{-\frac{2}{a\sqrt{x}}} \left\{ -\frac{A}{a} - \frac{C}{a} \int \frac{dx}{x} e^{\frac{2}{a\sqrt{x}}} \right\}$$

and $$y = \overline{|-D} \frac{e^{-\frac{2}{a\sqrt{x}}}}{\sqrt{x}} \left\{ -\frac{A}{a} - \frac{C}{a} \int \frac{dx}{x} e^{\frac{2}{a\sqrt{x}}} \right\} \quad \ldots \text{(1.)}$$

$$= \int_0^\infty e^{-px} \frac{dp}{\sqrt{p}} e^{-\frac{2\sqrt{p}}{a}} \left\{ -\frac{A}{a} + \frac{C}{a} \int \frac{dp}{p} e^{-\frac{2\sqrt{p}}{a}} \right\} \quad \ldots \text{(2.)}$$

The constants A and C are not both of them arbitrary. In fact, it will be seen that $C = \frac{A}{\overline{|0}}$, so that the final form of equation (1.) is

$$y = -\frac{A}{a} \overline{|-D} \frac{e^{-\frac{2}{a\sqrt{x}}}}{\sqrt{x}} \left\{ 1 + \frac{1}{\overline{|0}} \int \frac{dx}{x} e^{\frac{2}{a\sqrt{x}}} \right\}$$

Ex. 2. To solve the equation

$$\frac{dy}{dx}-a\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}-\frac{1}{2}\frac{y}{x}=\mathrm{X}.$$

1. By the method employed at p. 269 of my previous Memoir, this equation becomes

$$(-\mathrm{D}+\tfrac{1}{2})y+a(-1)^{\frac{1}{2}}\frac{\overline{-\mathrm{D}+1}}{|{-\mathrm{D}+\tfrac{1}{2}}}e^{\frac{\theta}{2}}y=-\mathrm{X}x$$

or

$$\frac{y}{x}-a\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\cdot\frac{y}{x}=\frac{1}{x}\cdot\frac{\mathrm{X}x}{\mathrm{D}-\frac{1}{2}}$$

$$=\frac{1}{\sqrt{x}}\int\frac{\mathrm{X}\,dx}{x^{\frac{1}{2}}}$$

$$=\mathrm{P},\text{ suppose;}$$

then

$$\frac{y}{x}=\frac{1}{\dfrac{d}{dx}-a^2}\left(\frac{d\mathrm{P}}{dx}+a\frac{d^{\frac{1}{2}}\mathrm{P}}{dx^{\frac{1}{2}}}\right)$$

$$y=\mathrm{A}xe^{a^2x}+xe^{a^2x}\int e^{-a^2x}dx\left(\frac{d\mathrm{P}}{dx}+a\frac{d^{\frac{1}{2}}\mathrm{P}}{dx^{\frac{1}{2}}}\right)$$

2. By Professor BOOLE's method, given in the *Philosophical Magazine* for February 1847. The given equation when written

$$x(d-a\,d^{\frac{1}{2}})y-\tfrac{1}{2}y=\mathrm{X}x;$$

may be thrown into the form

$$f(d)\big(x\,\mathrm{F}(d)\big)y=\mathrm{X}x;\text{ provided}$$

$$f(d)\,\mathrm{F}(d)=d-a\,d^{\frac{1}{2}},\text{ and}$$

$$f'(d)\,\mathrm{F}(d)=-\tfrac{1}{2}.$$

We have, therefore, $f(d)=(d^{\frac{1}{2}}-a)^{-1}$

$$\mathrm{F}(d)=(d^{\frac{1}{2}}-a)^2\,d^{\frac{1}{2}}$$

and

$$y=d^{-\frac{1}{2}}(d^{\frac{1}{2}}-a)^{-2}\{x^{-1}(d^{\frac{1}{2}}-a)\mathrm{X}x\}$$

3. By Mr HARGREAVE's method, given in the *Philosophical Transactions* for 1848, p. 31.

By changing d into x, and x into $-d$ in the equation

$$x(d-a\,d^{\frac{1}{2}})y-\tfrac{1}{2}y=\mathrm{X}x\quad\ldots\ldots\;(1.)$$

it becomes

$$d(x-a\,x^{\frac{1}{2}})y+\tfrac{1}{2}y=d\mathrm{X}\quad\ldots\ldots\;(2.)$$

which, being an ordinary linear equation, gives, as its solution, if we write d^{-1} for $\int dx$,

$$y=x^{-\frac{1}{2}}(x^{\frac{1}{2}}-a)^{-2}d^{-1}\{(x^{\frac{1}{2}}-a)\,d\mathrm{X}\}\quad\ldots\ldots\;(3.)$$

Now, since equation (2.) has been derived from equation (1.) by the change of d into x, and x into $-d$, equation (1.) may be derived from equation (2.) by changing x into d and d into $-x$. Consequently (in some cases, at least, of which this form is an instance) the *solution* of equation (1.) may be derived from the *solution* of equation (2.) by the same change. Hence the solution of the given equation is

$$y=d^{-\frac{1}{2}}(d^{\frac{1}{2}}-a)^{-2}\{x^{-1}(d^{\frac{1}{2}}-a)\,x\,\mathrm{X}\},$$

which is precisely the solution given by Professor BOOLE's process.

We shall conclude the present Memoir by comparing the methods here exhibited in two particular cases of the second example.

CASE 1. Let $\mathrm{X}=0$; then the first method gives $y=\mathrm{A}\,x\,e^{a^2x}$: whilst the second and third methods give

$$y=d^{-\frac{1}{2}}(d^{\frac{1}{2}}-a)^{-2}\,.\,0.$$

Now, in a former paper (*Transactions of the Royal Society of Edinburgh*, vol. xiv., p. 252), I have shewn, in Cor. 1. to Example 3, that

$$(d^{\frac{1}{2}}-a)^{-2}\,.\,0=\mathrm{A}\,e^{a^2x}+\mathrm{B}\,x\,e^{a^2x}$$

Hence

$$y=d^{-\frac{1}{2}}\{\mathrm{A}\,e^{a^2x}+\mathrm{B}\,x\,e^{a^2x}\}$$

$$=\frac{\mathrm{A}}{a}\,e^{a^2x}+\frac{\mathrm{B}\,x}{a}\,e^{a^2x}-\frac{\mathrm{B}}{2\,a^3}\,e^{a^2x}$$

The condition $\mathrm{B}=2\,\mathrm{A}\,a^2$ reduces this solution to the former.

CASE 2. Let

$$\mathrm{X}=b\left(\frac{1}{x^{\frac{3}{2}}}+\frac{a\sqrt{-1}}{\sqrt{\pi}}\;\frac{1}{x}\right);$$

then, by the first method,

$$\mathrm{P}=\frac{1}{\sqrt{x}}\int\frac{\mathrm{X}\,dx}{\sqrt{x}}=\frac{b}{\sqrt{x}}\left\{-\frac{1}{x}-\frac{2\,a\sqrt{-1}}{\sqrt{\pi}}\;\frac{1}{\sqrt{x}}\right\}$$

$$=-\frac{b}{x^{\frac{3}{2}}}-\frac{2\,a\,b\sqrt{-1}}{\sqrt{\pi}}\;\frac{1}{x}$$

$$\frac{d\,\mathrm{P}}{d\,x}+a\,\frac{d^{\frac{1}{2}}\,\mathrm{P}}{d\,x^{\frac{1}{2}}}=b\left(\frac{3}{2}\,\frac{1}{x^{\frac{5}{2}}}+\frac{a^2}{x^{\frac{3}{2}}}\right)$$

$$y=\mathrm{A}\,x\,e^{a^2x}+b\,x\,e^{a^2x}\int e^{-a^2x}\,dx\left(\frac{3}{2}\,\frac{1}{x^{\frac{5}{2}}}+\frac{a^2}{x^{\frac{3}{2}}}\right)$$

$$=\mathrm{A}\,x\,e^{a^2x}-\frac{b}{\sqrt{x}}$$

By the second and third methods,

$$\frac{1}{x}(d^{\frac{1}{2}}-a)\,\mathrm{X}\,x=b\left(\frac{\sqrt{-1}}{\sqrt{\pi}}\;\frac{1}{x^2}-\frac{a}{x^{\frac{3}{2}}}-\frac{a^2\sqrt{-1}}{\sqrt{\pi}\,x}\right)$$

Also

$$d^{-\frac{1}{2}}(d^{\frac{1}{2}}-a)^{-2}=\frac{d^{\frac{1}{2}}+2\,a+d^{-\frac{1}{2}}}{(d-a^2)^2}$$

$$\therefore\quad y=(d-a^2)^{-2}\,(d^{\frac{1}{2}}+2\,a+d^{-\frac{1}{2}})\,b\left(\frac{\sqrt{-1}}{\sqrt{\pi}}\;\frac{1}{x^2}-\frac{a}{x^{\frac{3}{2}}}-\frac{a^2\sqrt{-1}}{\sqrt{\pi}\,x}\right)$$

$$=(d-a^2)^{-2}\,b\left(-\frac{3}{4\,x^{\frac{5}{2}}}-\frac{a^2}{x^{\frac{3}{2}}}-\frac{a^4}{x^{\frac{1}{2}}}\right)$$

an ordinary linear differential equation, of which the solution is

$$y=\mathrm{A}\,x\,e^{a^2x}+\mathrm{B}\,e^{a^2x}-\frac{b}{\sqrt{x}}.$$

This agrees with the former solution by making $\mathrm{B}=0$.

III.—*On the Constitution of Codeine and its Products of Decomposition.* By THOMAS ANDERSON, M.D.

(Read 15th April 1850.)

During the last few years, great progress has been made in the study of the organic alkalies, and the discovery of methods by which these substances can be artificially produced, and the long train of investigations by which it has been followed, has greatly extended our previous information, and afforded us some definite ideas regarding their constitution. The advance made has, however, related entirely to the volatile bases produced by artificial processes, and our knowledge of the natural fixed alkaloids stands very much where it did some years since, and is still very imperfect, and in regard to many entirely fragmentary; so much so, indeed, that of all the alkaloids of this class described in chemical works, there are not perhaps a dozen of which the constitution can be considered as definitely fixed, and not half that number of which we know the products of decomposition. The fact is, that the interest attaching to the artificial bases has altogether diverted the attention of chemists from the natural alkalies, which have not hitherto proved a very productive field of inquiry; at least the researches to which many of them were subjected ten or fifteen years since, proved comparatively unfruitful in their results. The want of success which attended their investigation at that time, however, is attributable, partly to the imperfections of the method of analysis of such compounds, and partly to our entire ignorance of the constitution of the nitrogenous substances generally. Neither of these difficulties can now be said to exist; and the investigation of the volatile bases has so far elucidated the constitution of these substances generally, that we are now in the condition to return to the examination of the far more complex natural bases with some prospect of ultimate success. Chemists are, accordingly, beginning to turn their attention to this field of inquiry, and during the last few months, several investigations have been published, by which the constitution and products of decomposition of several important bases have been established; and in the present paper I propose communicating to the Royal Society the results of a series of investigations of codeine and its compounds, which has enabled me to add it to the number of those of which the constitution is definitely fixed.

It will be unnecessary for me to premise any observations regarding the history of codeine and its discovery, which are sufficiently well known, further than to refer to the analyses and formulæ given for it by the different chemists by whom it has been examined. Codeine has been analysed by its discoverer, Robiquet, and by Couerbe, Regnault, Will, Gregory, and Gerhardt. All the analyses of these observers I have brought together in the following table, in which, however, the *per centage* results are not those found in the original papers, but have been calculated from the analytical numbers according to the new equivalent of carbon.*

Anhydrous Codeine.

	Robiquet.†	Couerbe.‡		Regnault.§		Gregory.‖	Will.¶
Carbon,	70·363	71·59	72·10	73·31	72·93	73·18	73·27
Hydrogen,	7·585	7·12	7·17	7·19	7·23	7·23	7·25
Nitrogen,	5·353	5·23	...	4·89	4·89	4·82	...
Oxygen,	16·699	16·06	...	14·61	14·95	14·77	...
	100·00	100·00		100·00	100·00	100·00	

Crystallised Codeine.

	Gerhardt.**	
Carbon,	67·77	67·87
Hydrogen,	7·59	7·33
Nitrogen,	...	...
Oxygen,	...	...

From these analyses, four different formulæ have been deduced. Two of these, however, those of Robiquet and Couerbe, do not require particular mention, as they were unsupported by any accurate determination of the atomic weight of the substance, and are now certainly known not to represent its true constitution. That which has been hitherto most generally adopted by chemists is the one founded by Regnault upon his analysis, and represents codeine as $C_{35}\,H_{20}\,NO_5$, and the crystallised base as $C_{35}\,H_{20}\,NO_5+2\,HO$; the calculation of which gives

	Anhydrous.	Crystallised.
Carbon,	73·94	69·53
Hydrogen,	7·04	7·28
Nitrogen,	4·92	4·63
Oxygen,	14·10	18·50
	100·00	100·00

The analyses of Will and Gregory have usually been quoted in confirmation of this formula. It is clear, however, that the agreement between the calculated and experimental results is by no means satisfactory, either in them or in

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* In the case of Robiquet and Will's analyses, the details of the experiment are not given. I have, therefore, been obliged to convert the per centage of carbon into carbonic acid, according to the old equivalent of carbon, and recalculate it into carbon according to the new equivalent.

† Annales de Chimie et de Physique, vol. li., p 265.

‡ Ibid., vol. lix., p. 158. § Ibid., vol. lxviii., p. 136.

‖ Annalen der Chimie und Pharmacie, vol. xxvi., p. 44. ¶ Ibid.

** Revue Scientifique, vol. x., p. 203.

Regnault's own results; the highest amount obtained for the carbon being 0·63 per cent. below the calculation, while the lowest differs by more than one per cent., and the mean of the whole four gives 0·77 too little carbon, involving a loss which could not possibly have occurred in carefully made analyses.

Partly on account of this difference, and partly guided by his views regarding the divisibility of formulæ, **Gerhardt** was induced to doubt the exactitude of **Regnault**'s formula, which presents three different deviations from his law; the number of equivalents of carbon and of oxygen being uneven, and the sum of the equivalents of hydrogen and nitrogen also indivisible by two. He therefore repeated its analysis, using the crystallised codeine, and obtained the results contained in the table, and deduced from them the formula $C_{36} H_{21} NO_6$ for the anhydrous base, which gives the calculated results:

	Anhydrous.	Crystallised.
Carbon, . . .	72·24	68·13
Hydrogen, . . .	7·02	7·25
Nitrogen, . . .	4·68	4·41
Oxygen, . . .	16·06	20·11
	100·00	100·00

and tallies extremely well with his analysis. This formula has, however, been again called in question by **Dollfus**,* who has endeavoured to determine the constitution and atomic weight of the alkaloids by the analysis of their hydrosulphocyanates, and obtained from the codeine salt of that acid; results agreeing with the formula $C_{34} H_{19} NO_5$. Considering the known accuracy of **Regnault**, and of the chemists by whom his formula has been confirmed, I considered it an essential preliminary to my investigation to repeat its analysis with all possible care, so as to determine which of the two represents its true constitution.

I. *Preparation and Analysis of Codeine.*

I have little to add to the information we already possess regarding the preparation of codeine. I have obtained it, as usual, from the mother liquor from which morphia has been precipitated by ammonia. As the codeine forms only from a sixteenth to a thirtieth of the morphia, it is, of course, mixed in this fluid with a corresponding quantity of muriate of ammonia, which must be decomposed by potash, in order to obtain it. Much advantage is gained, however, by first evaporating the fluid to crystallisation, and expressing the crystals deposited, as in this way the greater part of the muriate of ammonia, which is the more soluble salt of the two, is left in solution; and by repeating the crystallisation many times, it may be entirely removed, and crystals obtained which are pure hydrochlorate of codeine. For the preparation of codeine, however, it would be worse than useless to carry the process thus far, as the solubility of hydrochlorates of codeine and ammonia differs so little that much of the former salt would be lost; but by carrying it a certain length, the greater part of the sal-ammoniac may be separated without any material loss of codeine, and the subsequent steps of the process much facilitated. The crystals so obtained being dissolved in boiling water, strong solution of caustic potash is added in excess, when codeine is in part precipitated as an oil, which by-and-by concretes into a solid mass, and is partly deposited in crystals as the solution cools. By evaporating the fluid, another crop of crystals is obtained; and, finally, when the mother-liquor has been concentrated to a very small bulk, it becomes filled on cooling with long silky needles of morphia, which has been retained in solution by the excess of potash. A certain quantity of morphia appears always to remain in solution along with the codeine; at least I have found it in all the mother-liquors I have examined, although its quantity appears to vary considerably. Its presence in this solution has been observed before, and it has been stated that it exists in the form of a double salt with codeine; this, however, is not consistent with my own experience, at least the salt separated from the muriate of ammonia by successive crystallisations contained no morphia, but, as has been already stated, was pure hydrochlorate of codeine.

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* Annalen der Chimie und Pharmacie, vol. lxv., p. 218.

The crystals of codeine precipitated by potash, in the manner described, are always more or less coloured. They are purified by solution in hydrochloric acid, boiling with animal charcoal, and reprecipitation with a slight excess of potash, and the precipitate obtained finally dissolved in ether, to separate any morphia which may adhere to it. For this purpose hydrous ether is best adapted; and it ought to be free from alcohol, as if any be present, the ether evaporates, and a syrupy fluid is left behind, which refuses to crystallise. When the ether is anhydrous, it dissolves codeine with much greater difficulty, and by evaporation small crystals are deposited, which are anhydrous.

The codeine employed for analysis was dried at 212°. The three first were made with codeine crystallised from hydrous ether, which lost two equivalents of water at 212°; the last was anhydrous codeine in small colourless crystals.

I. 6·120 grains of codeine, with oxide of copper, gave
16·135 ... of carbonic acid, and
3·888 ... of water.

II. 5·896 grains of codeine, with oxide of copper, gave
15·616 ... of carbonic acid, and
3·737 ... of water.

III. 4·688 grains of codeine, with chromate of lead, gave
12·392 ... of carbonic acid, and
3·015 ... of water.

IV. 5·858 grains of codeine, with chromate of lead, gave
15·485 ... of carbonic acid, and
3·780 ... of water.

5·395 grains of codeine gave, by **Varrentrap** and **Will**'s method, 3·79 grains of ammonio-chloride of platinum.

5·898 grains gave, by the same method, 4·32 grains of ammonio-chloride of platinum.

	I.	II.	III.	IV.
Carbon, . . .	71·91	72·02	72·09	72·09
Hydrogen, . .	7·05	7·04	7·14	7·16
Nitrogen, . .	4·41	4·60	4·50	...
Oxygen, . .	16·63	16·34	16·27	...
	100·00	100·00	100·00	

These results confirm, in all respects, the formula $C_{36} H_{21} NO_6$, the calculated results of which are given on a former page. The determination of the atomic weight of codeine by the analysis of its platinum salt, presented considerable difficulties, and at first gave extremely discordant results, the per centage of platinum varying from 18·51 to 20·30. I found, however, that by precipitating in the cold, a salt was obtained, to be afterwards described, which gave sufficiently uniform results. This salt, dried at 212°, retained an equivalent of water. It gave, as the mean of seven experiments, the details of which will be afterwards given, 19·25 per cent. of platinum, while the calculation, according to the above formula, requires 19·19 per cent. These determinations leave no doubt as to the formula of codeine; and they are fully confirmed by the result of the analyses of the substances to be described in the sequel of this paper.

Codeine crystallised from water or hydrous ether is obtained in crystals, often of considerable size, belonging to the right-prismatic system, and presenting a considerable number of modifications. These crystals contain two equivalents of water of crystallisation, as determined by this experiment:—

7·126 grains crystallised codeine lost, at 212°, 0·454=5·66 per cent. water.

The calculated result gives 5·67.

Codeine is an extremely powerful base, rapidly restoring the blue of reddened litmus, and precipitating oxides of lead, copper, iron, cobalt, nickel, and other metals, from their solutions. It is precipitated by potash from its salts; and is generally stated to be insoluble in that alkali, but this is true only of very highly concentrated solutions, as a considerable quantity of strong potash may be added to a saturated solution of codeine in water without producing precipitation; and even when a very large amount of potash is added, a certain quantity of the base is still retained in solution. Codeine is soluble in ammonia, but not more so than in water. 100 parts of a moderately strong solution of ammonia dissolved, at 60°, 1·46 parts of codeine; and according to **Robiquet**, 100 parts of water, at 59°, dissolve 1·26 parts. Contrary to what is usually stated, I have found that codeine is precipitated from all its salts by ammonia; it does not, however, fall immediately, but is slowly deposited in small transparent crystals.

II. *Salts of Codeine.*

Hydrochlorate of Codeine.—This salt is readily obtained by saturating hot dilute hydrochloric acid with pure codeine. If the solution has been sufficiently concentrated, it becomes nearly solid on cooling, but if more dilute, the salt is deposited in radiated groups of short needles, which, under the microscope, are found to be four-sided prisms terminated by dihedral summits. It is never obtained in large crystals, even when considerable quantities are crystallised. These crystals are soluble in 20 times their weight of water at 60°, and in less than their weight of water at 212°. Codeine is precipitated from the saturated cold solution immediately by potash; ammonia gives no precipitate, but after some time colourless crystals are deposited. The crystallised hydrochlorate of codeine contains water of crystallisation, and presents some curious anomalies in its relations to that fluid. When dried in the air, it retains four equivalents of water, one of which escapes at 212°, but the remaining three are only expelled at 250°, and at the same time the salt loses acid, and acquires an alkaline reaction. It would appear, also, that under certain circumstances, the salt is deposited in anhydrous crystals, as one analysis of it dried at 212°, gave numbers corresponding to the anhydrous salt. I could not, however, again succeed in obtaining it in this condition; but many analyses were made which gave results lying between those of the anhydrous and crystallised salts, and the only means of explaining the discrepancy is by supposing that the two sorts of crystals had been deposited simultaneously and in variable proportions. The following is the analysis of the salt dried at 212°:—

6·035 grains hydrochlorate of codeine gave
13·208 ... of carbonic acid,
3·830 ... of water.

	Experiment.	Calculation.		
Carbon, . .	59·68	59·58	C_{36}	216
Hydrogen, . .	7·08	6.89	H_{25}	25
Nitrogen, . .	...	3·86	N	14
Oxygen, . .	...	19·88	O_9	72
Chlorine, . .	...	9·79	Cl	35·5
		100·00		362·5

10·735 grains of the salt lost, at 212°, 0·31 grains of water=2·88 per cent. One equivalent of water gives by calculation 2·42 per cent. The formula of the air-dried salt is therefore $C_{36}\,H_{21}\,NO_6\,H\,Cl + 4\,HO$.

The anhydrous salt gave the following results. Of these, No. I. is the salt obtained by direct crystallisation from the morphia mother-liquor; No. II. is that which was got anhydrous at 212°; and No. III. is a portion dried at 250°; it had become strongly alkaline, which accounts for the excess of carbon obtained.

I. 6·171 grains dried at 250° gave
14·565 ... carbonic acid, and
3·795 ... water.

II. 4·286 grains dried at 212° gave
10·014 ... carbonic acid, and
2·603 ... water.

III. 5·740 grains dried at 250° gave
13·667 ... carbonic acid, and
3·467 ... water.

	Experiment.			Calculation.		
	I.	II.	III.			
Carbon,	64·37	64·56	64·93	64·38	C_{36}	216
Hydrogen,	6·83	6·74	6·71	6·55	H_{22}	22
Nitrogen,	...	...	...	4·17	N	14
Oxygen,	...	...	...	14·32	O_6	48
Chlorine,	...	...	...	10·58	Cl	35·5
				100·00		335·5

These results correspond to the formula $C_{36} H_{21} NO_6 H Cl$.

Hydriodate of Codeine is obtained by dissolving codeine in hot hydriodic acid, and allowing the solution to cool. It is deposited in long slender needles, which fill the whole fluid, if it have been sufficiently concentrated. It is of rather sparing solubility in cold water, requiring about 60 times its weight, but is much more soluble in boiling water. Its saturated cold solution is precipitated by ammonia on standing. No difficulty was experienced in its analysis.

I. 6·336 grains hydriodate, dried at 212°, gave
11·190 ... of carbonic acid, and
3·247 ... of water.

II. 5·801 grains dried at 212°, gave
10·347 ... of carbonic acid, and
2·977 ... of water.

5·733 grains of hydriodate of codeine gave 2·994 grains of iodide of silver.

	Experiment.		Calculation.		
	I.	II.			
Carbon,	48·16	48·64	48·60	C_{36}	216
Hydrogen,	5·69	5·70	5·40	H_{24}	24
Nitrogen,	...	...	3·15	N	14
Oxygen,	...	...	14·45	O_8	64
Iodine,	...	28·22	28·40	I	126·36
			100·00		444·36

The formula of the salt, dried at 212°, is, therefore, $C_{36} H_{21} NO_6 HI + 2 HO$.

Sulphate of Codeine.—Crystallises in radiated groups of long needles, or by spontaneous evaporation in flattened four-sided prisms. It requires for solution 30 times its weight of cold water, but it is very soluble in the heat. When pure, it is neutral to test paper, but it is very liable to retain a small quantity of acid, which can be got rid of by repeated crystallisations. The first analysis was made with the salt which had been only once crystallised, and has therefore given an excess of sulphuric acid.

Analysis of the salt, dried at 212°, gave the following results:—

I. 5·564 grains sulphate of codeine gave
12·536 ... of carbonic acid, and
3·270 ... of water.

II. 5·677 grains of sulphate of codeine gave
12·831 ... of carbonic acid,
3·324 ... of water.

I. 9·540 grains of sulphate of codeine gave
3·265 ... of sulphate of baryta.

II. 10·826 grains of sulphate of codeine gave
3·650 ... of sulphate of baryta.

	Experiment.		Calculation.		
	I.	II.			
Carbon,	61·44	61·64	62·07	C_{36}	216
Hydrogen,	6·52	6·50	6·39	H_{22}	22
Nitrogen,	...	...	4·03	N	14
Oxygen,	...	...	16·03	O_7	48
Sulphuric Acid,	11·75	11·54	11·49	SO_3	40
			100·00		348

27·13 grains of the crystallised salt lost, at 212°, 3·068 grains of water = 11·30 per cent. This corresponds to 5 equivalents of water, the calculated result for which is 11·45.

The formula of the crystallised salt is therefore

$$C_{36}H_{21}NO_6 HO SO_3 + 5 HO.$$

Nitrate of Codeine.—Is obtained by slowly adding nitric acid, of specific gravity 1·060, to powdered codeine, an excess of nitric acid being carefully avoided, as the base is rapidly decomposed by it with the formation of a product of substitution to be afterwards described. The nitrate is readily soluble in boiling water, and is deposited on cooling in small prismatic crystals. Heated on platinum, it melts, and on cooling, concretes into a brown resinous mass; at a higher temperature it is rapidly decomposed, leaving a bulky coal, difficult of incineration.

6·360 grains of nitrate of codeine, dried at 212°, gave
13·854 ... of carbonic acid,
3·746 ... of water.

These results correspond with the formula $C_{36}H_{21}NO_6 HO NO_5$.

	Experiment.	Calculation.		
Carbon, ...	59·40	59·66	C_{36}	216
Hydrogen, . . .	6·54	6·07	H_{22}	22
Nitrogen, ...	...	7·73	N_2	28
Oxygen, ...	...	26·54	O_{12}	96
		100·00		362

Phosphate of Codeine.—Several phosphates of codeine appear to exist, but I have only examined that which is obtained by saturating tribasic phosphoric acid with codeine in powder. In this way a fluid is obtained, which, when concentrated to a small bulk, refuses to crystallise, but from which crystals are immediately precipitated by the addition of strong spirit. The salt is thus obtained in small scales, or in short thick prisms. It is readily soluble in water, and cannot be obtained in crystals from the solution. Its analysis gave the following results, corresponding with the formula $C_{36}\,H_{21}\,NO_6\,HO\,2\,HO\,PO_5$.

6·343 grains phosphate of codeine, dried at 212°, gave
12·618 ... of carbonic acid,
3·708 ... of water.

	Experiment.	Calculation.		
Carbon, ...	54·25	54·27	C_{36}	216
Hydrogen, . . .	6·49	6·03	H_{24}	24
Nitrogen, ...	...	3·52	N	14
Oxygen, ...	...	18·09	O_9	72
Phosphoric acid, . .	...	18·09	PO_5	72
		100·00		398

6·911 grains of the crystallised salt lost, at 212°, 0·434 grains of water = 6·27 per cent. Three equivalents of water correspond to 6·35 per cent.; and the formula of the crystallised salt is, consequently, $C_{36}\,H_{21}\,NO_6\,HO\,2\,HO\,PO + 3\,HO$.

Oxalate of Codeine.—This salt is deposited, on cooling its saturated hot solution, in short prisms, and sometimes in scales. It requires 30 times its weight of water at 60° for solution, and about half its weight at 212°. Heated to 212° it loses water of crystallisation; at 250° it becomes brown, and at a higher temperature it is entirely decomposed.

6·073 grains oxalate of codeine, dried at 212°, gave
14·739 ... of carbonic acid,
3·608 ... of water.

	Experiment.	Calculation.		
Carbon, ...	66·19	66·28	C_{38}	228
Hydrogen, . . .	6·60	6·39	H_{22}	22
Nitrogen, ...	...	4·07	N	14
Oxygen, ...	...	23·26	O_{10}	80
		100·00		344

10·050 grains of the crystallised oxalate lost, at 212°, 0·704 grains of water = 7·00 per cent., corresponding to three equivalents of water, which requires 7·27 per cent. The formula of the crystallised salt is, therefore, $C_{36}\,H_{21}\,NO_6\,HO\,C_2\,O_3+3\,HO$.

Hydrosulphocyanate of Codeine.—This salt has been already examined by Dollfus;* but I have prepared it, and repeated the analysis, with results differing somewhat from those obtained by him. It is readily obtained by mixing solutions of hydrochlorate of codeine and of sulphocyanide of potassium, and is slowly deposited in beautiful radiated needles.

6·164 grains of hydrosulphocyanate, dried at 212°, gave
14·285 ... carbonic acid, and
3·543 ... water.

7·444 grains, burnt with nitre and carbonate of soda, gave 4·899 grains of sulphate of baryta.

These results correspond with the formula $C_{36}\,H_{21}\,NO_6\,H\,C_2N\,S_2$, as is shewn by the following per centage calculation, to which I have added the results obtained by Dollfus:—

	Experiment.		Calculation.		
	Dollfus.				
Carbon, . . .	62·30	63·20	63·68	C_{38}	228
Hydrogen, . . .	6·13	6·38	6·14	H_{22}	22
Nitrogen, . . .	...	...	7·82	N_2	28
Oxygen, . . .	...	...	13·43	O_6	48
Sulphur, . . .	...	9·04	8·93	S_2	32
			100·00		358

11·613 grains of the crystallised salt, dried at 212°, lost 0·288 grains of water = 2·47 per cent., corresponding to one equivalent of water, the calculation of which gives 2·45 per cent.

In the analysis of Dollfus, there is manifestly a loss of carbon, as the results are quite incompatible with those of the base and its other salts. In the same paper Dollfus has also determined the amount of sulphocyanogen by precipitation with silver, and the results obtained agree better with the formula given above than with his own.

Chloride of Platinum and Codeine.—When bichloride of platinum is added to a moderately concentrated solution of hydrochlorate of codeine, a pale-yellow, pulverulent precipitate is deposited. If this be allowed to stand for some time in the solution, or still better, if it be collected on a filter and kept moist, it begins to change in its appearance; specks of darker colour appear in it, and it is gradually converted into a mass of crystalline grains of an orange-yellow colour. The fluid which filters off deposits, on standing, a small quantity of larger grains.

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* Annalen der Chimie und Pharmacie, vol. lxv., p. 218.

The change which takes place in this manner is not always complete, and the granular crystals are often mixed with unchanged yellow powder. When the chloride of platinum is added to a more dilute solution of hydrochlorate of codeine, precipitation does not take place immediately, but in a short time the salt is deposited in minute tufts of silky needles. The salt is soluble in boiling water, and is deposited on cooling partly in grains, partly as a powder. By this process, however, it is partially changed; and I have ascertained that by ebullition, with excess of chloride of platinum, it is completely decomposed. I have not as yet, however, followed up this observation. I at first attempted to purify the salt by solution in water and alcohol, in which it is also soluble; and a number of analyses were made, which gave extremely contradictory results; but by precipitation in the cold, and without excess of platinum, sufficiently uniform results were obtained.

When dried at 212°, the salt retains an equivalent of water, which is expelled at 250°, but not without occasioning partial decomposition of the substance, which evolves acid, and acquires a brownish colour. The following are the results of analysis:—

I. { 7·240 grains of platinum salt, dried at 212°, gave
11·072 ... of carbonic acid, and
2·925 ... water.

II. { 9·394 grains of platinum salt gave
14·593 ... of carbonic acid, and
3·912 ... of water.

III. { 7·648 grains of platinum salt gave
11·694 ... of carbonic acid, and
3·450 ... of water.

IV. { 6·665 grains of platinum salt gave
10·230 ... of carbonic acid, and
2·835 ... of water.

V. { 7·383 grains of platinum salt gave
11·372 ... of carbonic acid, and
3·304 ... of water.

7·247	grains platinum salt gave	1·400	grains platinum,	=19·31	per cent.
10·030	... ...	1·920	...	=19·14	...
9·775	... ...	1·850	...	=18·92	...
10·471	... ...	2·020	...	=19·32	...
8·428	... ...	1·600	...	=18·98	...
6·790	... ...	1·296	...	=19·08	...
5·052	... ...	0·960	...	=19·00	...

	I.	II.	III.	IV.	V.	VI.	VII.
Carbon, .	41·70	42·36	41·70	41·80	42·00	...	...
Hydrogen, .	4·49	4·62	5·01	4·72	4·97	...	...
Nitrogen, .	...	...	...	...	...	...	...
Oxygen, .	...	...	...	...	...	...	...
Chlorine, .	...	...	...	...	...	...	...
Platinum, .	19·31	19·14	18·92	19·32	18·98	19·08	19·00

These analyses correspond with the formula

$$C_{36}H_{21}NO_6HCl+PtCl_2+HO.$$

of which the following is the calculated result compared with the mean of experiment:—

	Mean.	Calculation.		
Carbon, . . .	41·91	42·07	C_{36}	216·
Hydrogen, . . .	4·76	4·47	H_{23}	23·
Nitrogen, . . .	...	2·72	N	14·
Oxygen, . . .	...	10·94	O_7	56·
Chlorine, . . .	...	20·61	Cl_3	106·5
Platinum, . . .	19·25	19·19	Pt	98·7
		100·00		514·2

The air-dried salt gave the following results, when dried at 212°:—

14·845 grains lost 0·770 grains of water, =5·11 per cent.
14·546 ... 0·758 ... =5·20 ...

This corresponds to three equivalents of water, the calculated result for which gives 4·99 per cent. The crystallised salt is therefore represented by the formula

$$C_{36}H_{21}NO_6HCl+PtCl_2+4HO.$$

Codeine forms many other crystallisable salts, none of which, however, have been examined. The chromate is easily obtained in fine yellow needles. With solution of bichloride of mercury, codeine gives a white precipitate, soluble in boiling water and alcohol, and deposited on cooling in stellated groups of crystals. With chloride of palladium a yellow precipitate is obtained, which is decomposed by boiling, with separation of metallic palladium. Tartrate and hydrocyanate of codeine are uncrystallisable.

PRODUCTS OF DECOMPOSITION OF CODEINE.

III. *Action of Sulphuric Acid.*

Amorphous Codeine.—When codeine is dissolved in an excess of moderately-concentrated sulphuric acid, and the mixture digested on the sand-bath, the fluid gradually acquires a dark colour, and after some time gives a precipitate with carbonate of soda, which the salts of codeine are incapable of doing. The precipitate so obtained is codeine in a modified or amorphous condition, similar to that in which quinine is obtained by a similar treatment with excess of acid. By carefully regulating the temperature of the mixture of codeine and sulphuric acid, the amorphous codeine may be obtained in a state of purity; but it is neither so definite nor so stable a substance as quinoidine. After the action has been prolonged for some time, carbonate of soda is added to the fluid, and the gray precipitate obtained, collected on a filter, washed with water, dissolved in alcohol, and precipitated from the solution by means of water. As thus obtained, it is a gray powder, with a more or less green shade, insoluble in water, readily soluble in alcohol, and precipitated by ether from the solution. It fuses at 212° into a black resinous mass. In acids it is readily soluble, with the formation of salts which are amorphous, and dry up by evaporation into brown resins.

Analysis gave the following results:—

I.	5·400	grains amorphous codeine gave
	14·240	... of carbonic acid, and
	3·663	... of water.
II.	4·532	grains amorphous codeine gave
	12·054	... of carbonic acid, and
	2·781	... of water

	Experiment.		Calculation.
	I.	II.	
Carbon, ...	71·92	72·53	72·24
Hydrogen, ...	7·53	6·84	7·02
Nitrogen, ...	...	...	6·68
Oxygen, ...	...	...	16·06
			100·00

These results correspond sufficiently closely with those of codeine to shew that this substance is represented by the same formula. At the same time it is to be observed, that the action does not stop at the point at which amorphous codeine is formed; for the excess of carbon and deficiency of hydrogen in the second analysis (which occurred also in another analysis from a different preparation), appear to me to shew that some farther change had taken place. Indeed, by continuing the action of sulphuric acid, a deep-green powder was obtained, which contained sulphur, and agreed in its general properties with the sulphomorphide described by ARPPE, and the corresponding sulphonarcotide of LAURENT and GERHARDT.

IV. *Action of Nitric Acid.*

Nitrocodeine.—When strong nitric acid is poured upon codeine, and heat applied, violent action takes place, nitrous fumes are abundantly evolved, and the solution acquires a red colour. If the fluid be evaporated on the water-bath, a yellow resinous acid is left, which dissolves in ammonia and potash solutions, with a red colour.[*] If the nitric acid be employed in a sufficiently dilute state, a different result is obtained, and a nitrobase is formed, to which I give the name of nitrocodeine.

The preparation of this substance is a matter of some nicety, as by the continued action even of very dilute nitric acid it is rapidly destroyed. The operation succeeds best when the acid employed is of a specific gravity of 1·060. Acid of this density is heated in a flask, but not to ebullition, and finely-powdered codeine is added, and a moderate heat is sustained. In the course of a few minutes a small quantity of the fluid is poured out into a glass, and an excess of ammonia added; if no precipitate appears, the heat is kept up for a short time longer, and another quantity is then taken out and tested; and this is repeated until the precipitate, which makes its appearance when the acid is neutralised, ceases to increase. The fluid is then immediately saturated with ammonia, and stirred rapidly, when it becomes filled with a bulky precipitate of nitrocodeine. The action which takes place is extremely rapid, and the whole operation is complete in a few minutes; so that the experimenter requires to be carefully on the watch, in order to hit the right moment for precipitating the fluid. No red fumes are evolved; if they are seen, it is a sure sign that the action has gone too far, and that part of the codeine has been converted into the resinous acid already mentioned. On this account it is better to stop the action before the whole of the codeine is decomposed, the quantity left being easily recovered from the solution; but even with the greatest possible care, the formation of a small quantity of the resinous acid cannot be avoided, and its presence is always indicated by the dark colour which the fluid acquires when saturated by ammonia.

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[*] The constitution and properties of this substance will be detailed in a future communication.

On the addition of ammonia, the nitrocodeine falls in the form of minute silvery plates, with a very slight shade of yellow. It is purified by solution in hydrochloric acid, boiling with animal charcoal and a reprecipitation with ammonia, in order to separate colouring matter and any unchanged codeine which may have been precipitated along with the first crystals. The nitrocodeine is then crystallised by dissolving in dilute alcohol, or a mixture of alcohol and ether.

Nitrocodeine crystallised from alcohol is deposited in the form of slender silky needles of a pale fawn-colour, which, on drying, mat together into a silky mass. From alcohol and ether it is obtained by spontaneous evaporation in small yellowish crystals, which, under the microscope, are seen to be four-sided prisms, terminated by dihedral summits. Nitrocodeine is sparingly soluble in boiling water, from which it is deposited in minute crystals on cooling. It dissolves abundantly in boiling alcohol, and but sparingly in ether. It is soluble in acids, with the formation of salts which are neutral to test-paper, and from which potash and ammonia precipitate the base as a crystalline powder. When heated carefully, it melts into a yellow fluid, which concretes on cooling into a highly-crystalline mass. At a higher temperature, it suddenly decomposes without flame, leaving a bulky charcoal.

Its analysis yielded the following results, of which No. 1 is from the base crystallised from the first precipitate by ammonia, before I had observed its tendency to carry down codeine with it, and which has therefore given a slight excess; the others are from the pure base. Crystallised nitrocodeine is anhydrous.

I.	5·748	grains of nitrocodeine, dried at 212°, gave
	13·301	... of carbonic acid, and
	3·128	... of water.
II.	5·523	... of nitrocodeine gave
	12·724	... of carbonic acid, and
	2·887	... water.

III. { 4·463 grains of nitrocodeine gave
10·226 ... of carbonic acid, and
2·377 ... of water.

	Experiment.			Calculation.		
	I.	II.	III.			
Carbon, . . .	63·10	62·83	62·49	62·79	C_{36}	216
Hydrogen, . .	6·04	5·80	5·91	5·81	H_{20}	20
Nitrogen, . . .	...	...	...	8·11	N_2	28
Oxygen, . . .	...	...	...	23·29	O_{10}	80
				100·00		344

These results correspond with the formula $C_{36}H_{20}(NO_4)NO_6$, derived from that of codeine by the substitution of NO_4 in place of an equivalent of hydrogen. It is confirmed by the analysis of its platinum salt, which was found to contain 17·88 per cent. of platinum, giving for the atomic weight of the base 345·8: the calculated atomic weight is 344.

Hydrochlorate of Nitrocodeine.—Nitrocodeine dissolves readily in hydrochloric acid, and the solution on evaporation leaves the hydrochlorate in the form of a resinous mass, which cannot be made to crystallise.

Sulphate of Nitrocodeine is obtained in a radiated group of short-pointed needles, which are neutral to test-paper, and very soluble in boiling water.

4·687 grains of the sulphate, dried at 212°, gave 1·383 sulphate of baryta.

This corresponds to the formula $C_{36}H_{20}(NO_4)NO_6\ HO\ SO_3$, which requires

	Experiment.	Calculation.		
Nitrocodeine, ...	...	...	...	344
Water, ...	...	...	...	9
Sulphuric acid, . . .	10·13	10·17	...	40
				393

Oxalate of Nitrocodeine.—Crystallises in beautiful yellow short prisms, readily soluble in water.

Platinochloride of Nitrocodeine.—This salt is precipitated from the solution of the hydrochlorate as a yellow powder, insoluble in water and alcohol. Its analysis gave the following results:—

{ 8·113 grains of platinochloride of nitrocodeine, dried at 212°, gave
11·635 ... of carbonic acid, and
2.987 ... of water.

9·392 grains, dried at 212°, gave 1·68 grains platinum.

	Experiment.	Calculation.		
Carbon, ...	39·11	39·25	C_{36}	215
Hydrogen, ...	4·09	3·81	H_{21}	21
Nitrogen, ...	...	5·08	N_2	28
Oxygen, ...	...	14·58	O_{10}	80
Chlorine, ...	...	19·35	Cl_3	106·5
Platinum, ...	17·88	17·93	Pt	98·7
		100·00		550·2

8·670 grains of the precipitated salt, dried by long exposure to the air, lost, at 212°, 0·569 grains of water=6·56 per cent.

Four equivalents of water require 6·14 per cent. The formula of the salt is, therefore, $C_{36}\,H_{20}\,(NO_4)\,NO_6\,HCl+Pt\,Cl_2+4\,HO$.

When nitrocodeine dissolved in alcohol is treated with hydrosulphuret of ammonia in the water-bath, the solution gradually acquires a dark colour, and sulphur is deposited. When the action is complete, the filtered fluid gives with ammonia a brown amorphous precipitate, which, when dissolved in hydrochloric acid, and boiled with animal charcoal, gives, on precipitation, a pale-yellow base. The substance so obtained is very different from nitrocodeine; it is extremely soluble in alcohol, and is deposited from it as an amorphous powder. Once only did I obtain definite crystals, which were brownish rhomboids, but in too small quantity to admit of examination. The amorphous base did not give satisfactory results; and as its preparation is extremely troublesome, I did not pursue its investigation further. Arguing from what we know of the other bases formed by the same process, its constitution ought to be $C_{36}\,H_{22}\,N_2\,O_6$, and it might be called azocodeine.

V. *Action of Bromine on Codeine.*

Bromocodeine.—In order to obtain this substance, bromine-water is added in small successive portions to finely-powdered codeine. The base is rapidly dissolved, and the solution loses its colour of bromine, but acquires a peculiar and characteristic red shade. After a certain quantity of bromine has been added, small crystals make their appearance, which are hydrobromate of bromocodeine; but these are only observed if the bromine-water has been thoroughly saturated, and are deposited in small quantity only, the remainder being retained in solution. When the whole of the codeine has been got into solution, ammonia is added, and bromocodeine is immediately thrown down as a silvery-white powder. In this state it contains a small quantity of unchanged codeine. It is collected on a filter; washed several times with cold water, and redissolved in hydrochloric acid, from which it is reprecipitated by ammonia, and finally crystallised from boiling spirit. Bromocodeine is scarcely soluble in cold water; but by boiling, a somewhat larger quantity is taken up, and deposited again on cooling in minute prisms, terminated by dihedral summits. It is readily soluble in alcohol, particularly on boiling, and is best crystallised from spirit diluted with its bulk of water. The crystals in which it is deposited are always very small, but brilliantly white. It is scarcely soluble in ether. Exposed to heat, it melts into a colourless fluid, which is destroyed at a temperature slightly above its melting point. It dissolves in cold sulphuric acid, and the solution when heated becomes dark coloured. It is attacked by nitric acid, but much less rapidly than codeine itself.

Considerable difficulty was experienced in getting it absolutely free from codeine; and the first of the following analyses has given an excess in the carbon:—

I. { 6·119 grains bromocodeine, dried at 212°, gave
12·941 ... of carbonic acid, and
3·000 ... of water.

II. { 5·940 grains bromocodeine gave
12·461 ... of carbonic acid, and
2·910 ... of water.

5·268 grains gave 2·663 grains of bromide of silver.

	Experiment.		Calculation.		
	I.	II.			
Carbon,	57·67	57·21	57·14	C_{36}	216
Hydrogen,	5·44	5·44	5·29	H_{20}	20
Bromine,	...	21·50	21·16	Br	80
Nitrogen,	...	...	3·70	N	14
Oxygen,	...	...	12·71	O_6	48
			100·00		378

The formula is therefore $C_{36}H_{20}BrNO_6$. Bromocodeine is capable of uniting with water in two different proportions, as appears by the determination of the loss by drying.

11·784 crystallised bromocodeine, lost at 212° 0·273, =2·32 per cent.
9·308 ... ... 0·623, =6·69 ...
7·707 ... ... 0·512, =6·64 ...

The first of these results corresponds exactly to one equivalent of water, the calculated result for which gives 2·32 per cent. The other two give three equivalents, for which the calculation is 6·66. I am unable now to recollect how the bromocodeine used in the first experiment was obtained, but my impression is, that it was prepared in exactly the same manner as the rest.

Hydrochlorate of Bromocodeine is obtained in radiated needles closely resembling those of hydrochlorate of codeine.

Hydrobromate of Bromocodeine.—The crystals, which have been mentioned as making their appearance during the preparation of bromocodeine, are this salt. It is sparingly soluble in cold water, readily soluble in boiling water, and is deposited from the solution in small prismatic crystals. It contains two equivalents of water which are not expelled at 212°.

{ 8·424 grains of hydrobromate, dried at 212°, gave
13·956 ... of carbonic acid, and
3·985 ... of water.

	Experiment.	Calculation.		
Carbon,	45·18	45·28	C_{36}	216
Hydrogen,	5·25	4·84	H_{23}	23
Bromine,	...	33·54	Br_2	160
Nitrogen,	...	2·93	N	14
Oxygen,	...	13·41	O_8	64
		100·00		477

The formula of the salt is therefore $C_{36}H_{20}BrNO_6HBr+2HO$.

Platinochloride of Bromocodeine is precipitated as a pale-yellow powder, insoluble in water and alcohol.

8·126 grains, dried at 212°, gave 1·380 grains platinum.

	Experiment.	Calculation.		
Carbon,	...	36·97	C_{36}	216
Hydrogen,	...	3·59	H_{20}	20
Bromine,	...	13·70	Br	80
Nitrogen,	...	2·39	N	14
Oxygen,	...	5·23	O_6	48
Chlorine,	...	18·23	Cl	106·5
Platinum,	16·98	16·89	Pt	98·7
		100·00		584·2

Tribromocodeine.—By continuing the addition of bromine water beyond the point at which bromocodeine is formed, a further action takes place, and a bright-yellow precipitate makes its appearance, which at first redissolves in the fluid, but after a time becomes permanent, and goes on gradually increasing until a very large quantity of bromine has been employed, when at length a point is reached at which no further precipitate is produced. If the solution be left till next day, however, bromine again causes a precipitate; and if it be added, as long as anything falls, and the solution be again left standing, another precipitate is produced identical in all respects with that before obtained, and this may be repeated day after day for a very considerable time. The yellow precipitate so obtained is the hydrobromate of tribromocodeine. It is collected on a filter, and washed with water, in which it is very sparingly soluble. In order to obtain the base, this substance is dissolved in dilute hydrochloric acid and ammonia added, when the tribromocodeine is immediately precipitated as a flocky powder, which is washed with water, and purified by solution in alcohol, and precipitation with water.

Tribromocodeine is thus obtained as a bulky white precipitate, perfectly amorphous, and when dry, more or less gray in its colour. It is insoluble in water and ether, but readily soluble in alcohol. It is sparingly soluble in hydrochloric acid in the cold, but much more so by boiling. In this process, however, it appears to undergo a partial decomposition, as a small quantity is always left insoluble. Heated on platinum foil it becomes brown, and is entirely decomposed at its melting point, leaving a coal difficult of incineration.

The tribromocodeine employed for analysis was purified by a second solution in alcohol, and precipitation by ether. It gave the following results:—

8·014 grains of tribromocodeine, dried at 212°, gave
11·665 ... of carbonic acid, and
2·645 ... of water.

3·55 grains of tribromocodeine gave 3·727 grains bromide of silver.

	Experiment.	Calculation.		
Carbon, ...	39·69	40·27	C_{36}	216
Hydrogen, ...	3·66	3·35	H_{18}	18
Bromine, ...	44·68	44·72	B_3	240
Nitrogen, ...	...	2·61	N	14
Oxygen, ...	...	9·00	O_6	48
		100·00		536

These results agree sufficiently well with the formula $C_{36}\,H_{18}\,Br_3\,O_6$ produced by the substitution of three equivalents of bromine; and this formula has been confirmed by the analysis of its platinum salt, which will be given below.

In such cases as have been hitherto examined, the substitution of three equivalents of bromine in a base, has entirely destroyed its basic properties, but tribromocodeine is still a base, though an extremely feeble one. Its salts are all sparingly soluble in water and amorphous; and as there is no possibility of ascertaining their purity, I have not pursued their investigation to any extent.

Hydrochlorate of Tribromocodeine.—It is obtained by dissolving the base in hot dilute hydrochloric acid, and is deposited on cooling as an amorphous powder.

Hydrobromate of Tribromocodeine.—This is the substance deposited during the preparation of tribromocodeine. It is a bright-yellow powder, perfectly amorphous, and very sparingly soluble in cold water. On boiling, however, a larger quantity is taken up, and deposited unchanged on cooling.

Its analysis gave the following results:—

I. { 7·501 grains hydrobromate, dried at 212°, gave
8·868 ... of carbonic acid, and
1·915 ... of water.

II. { 6·840 grains hydrobromate, from another preparation, gave
8·072 ... of carbonic acid, and
1·767 ... of water.

3·762 grains hydrobromate gave 4·865 grains bromide of silver.

	Experiment.		Calculation.		
	I.	II.			
Carbon, . .	32·24	32·18	32·84	C_{72}	432
Hydrogen, . .	2·83	2·86	2·96	H_{39}	39
Bromine, . .	...	55·03	54·75	Br_9	720
Nitrogen, . .	...	...	2·12	N_2	28
Oxygen, . .	...	...	7·33	O_{12}	96
			100·00		1315

These results approach most nearly to the formula:—

$$2\,(C_{36}\,H_{18}\,Br_3\,N\,O_6) + 3\,H\,Br.$$

They present, however, a certain deficiency, both in the carbon and hydrogen, and an excess in the bromine; but no other formula can be found at all approximating to the experimental numbers, and the recurrence of the results, in portions prepared at different times, leaves no doubt that this is their real constitution; and, in all probability, the error may be due to the salt retaining a small excess of hydrobromic acid. The constitution is therefore remarkable, and I am not aware of any similar salt having been before observed.

Platinochloride of Tribromocodeine.—Bichloride of platinum throws down from solution of tribromocodeine, in hydrochloric acid, this salt, in the form of a brownish-yellow powder soluble in water and alcohol.

5·142 grains of platinum salt, dried at 212°, gave 0·669 grains of platinum.

	Experiment.	Calculation.		
Carbon, . .	...	29·10	C_{36}	216
Hydrogen, . .	...	2·55	H_{19}	19
Bromine, . .	...	32·33	Br_3	240
Nitrogen, . .	...	1·88	N	14
Oxygen, . .	...	6·57	O_3	48
Chlorine, . .	...	14·34	Cl_3	106·5
Platinum, .	13·07	13·29	Pt	98·7
		100·00		742·2

I have reason to believe that the action of bromine upon codeine does not terminate with the production of the base now described; but its further action did not appear to afford any products of a sufficient interest to induce me to prosecute the investigation in this direction. There must also no doubt exist a *dibromocodeine*, $C_{36}\ H_{19}\ Br_2\ N\ O_6$, but I did not meet with it in the course of my experiments, and have not made any special attempts to obtain it.

VI. *Action of Chlorine upon Codeine.*

We might anticipate that the action of chlorine upon codeine should be exactly similar to that of bromine; but this is not the case, as in place of a simple and definite action complex products are immediately obtained. When a current of chlorine is passed through an aqueous solution of codeine, the fluid immediately acquires a brown colour, which soon becomes very deep, and eventually almost black. From this solution ammonia throws down an amorphous, resinous base. With chlorine-water the solution also becomes rapidly brown, and a similar precipitate is obtained. As there was no method of determining in either of these cases when the action was complete, I did not attempt to examine the product. I succeeded better, however, by the action of chlorate of potash, and obtained a base corresponding to bromocodeine.

Chlorocodeine.—For the preparation of chlorocodeine a sufficient quantity of codeine is dissolved in an excess of dilute hydrochloric acid, at the temperature of about 150° or 160°. Finely-powdered chlorate of potash is then added, and the solution agitated. In the course of a few minutes a small quantity of the fluid is tested with ammonia, in order to see whether a precipitate is formed; and the action is allowed to go on until this is obtained, and the chlorocodeine is then precipitated by a slight excess of ammonia. The successful performance of this experiment requires exactly the same precautions as the preparation of nitrocodeine; and, unless the action is stopped at the right moment, further products of decomposition are obtained. The reaction which takes place is represented by this equation:—

$$3\,(C_{36}\,H_{21}\,NO_{36}\,H\,Cl) + 3\,H\,Cl + KO\,Cl\,O_5 = K\,Cl + 6\,HO + 3\,(C_{36}\,H_{20}\,Cl\,NO_6\,H\,Cl).$$

The chlorocodeine is precipitated in the form of a silvery crystalline powder, closely resembling bromocodeine; it has generally a yellowish colour, and the fluid from which it has deposited is coloured dark-red by the presence of a small quantity of some products of the further action of chlorine. It retains also a small quantity of codeine, from which it is purified by dissolving in hydrochloric acid, boiling with animal charcoal, and reprecipitating with ammonia; and it is finally obtained in crystals from its solution in boiling spirit.

In its general properties chlorocodeine closely resembles bromocodeine; so much so, indeed, that they may be easily confounded with one another. It is sparingly soluble in boiling water, and deposited, on cooling, in minute prisms exactly similar to, and apparently isomorphous with, those of bromocodeine. It is readily soluble in strong alcohol, especially with heat, and sparingly soluble in ether. It dissolves in sulphuric acid in the cold without change, but the solution is charred by heating. Nitric acid dissolves it, and the solution is decomposed by boiling, but not by any means so readily as codeine. Red fumes are evolved along with a peculiar and excessively pungent vapour.

Analysis gave the following results:—

I. 6·425 grains of chlorocodeine, dried at 212°, gave
15·315 ... of carbonic acid, and
3·601 ... of water.

II. 6·162 grains of chlorocodeine gave
14·597 ... of carbonic acid, and
3·372 ... of water.

5·030 grains of chlorocodeine gave 2·100 grains chloride of silver.

	Experiment.		Calculation.		
	I.	II.			
Carbon, . . .	65·00	64·62	64·76	C_{36}	216
Hydrogen, . .	6·22	6·08	5·99	H_{20}	20
Chlorine, . .	...	10·32	10·64	Cl	35·5
Nitrogen, . .	...	...	4·19	N	14
Oxygen, . .	...	...	14·42	O_6	48
			100·00		333·5

The crystallised base contains water which is expelled at 212°.

7·67 grains chlorocodeine lost 0·551 grains water, =7·18 per cent.
9·82 ... ... 0·740 ... =7·53 ...

The calculated number for three equivalents of water is 7·48 per cent.; and the formula of the crystallised base is therefore $C_{36} H_{20} Cl NO_6 + 3 HO$.

The salts of chlorocodeine are exactly similar in their properties to those of bromocodeine; so much so, that I have not thought it necessary to examine more than one or two of them.

Hydrochlorate of Chlorodeine.—Crystallises in groups of needles, readily soluble in water.

Sulphate of Chlorocodeine is deposited from its hot solution in radiated groups of short prisms, which dissolve abundantly in boiling water and alcohol.

10·874 grains of the crystallised salt, dried at 212°, gave 0·953 grains of water, and 3·078 grains of sulphate of baryta.

	Experiment.	Calculation.		
Chlorocodeine,	79·34	79·63	Base.	333·5
Sulphuric acid,	11·90	11·75	$HOSO_3$	49·0
Water,	8·76	8·662	4 HO	36·0
	100·00	100·00		418·5

Platinochloride of Chlorocodeine is obtained in the usual way, as a pale-yellow precipitate scarcely soluble in water. Its analysis gave the following results:—

7·212 grains platinochloride, dried at 212°, gave
10·658 ... of carbonic acid, and
2·655 ... of water.

8·793 grains platinochloride gave 1·608 grains platinum.

	Experiment.	Calculation.		
Carbon,	40·30	40·02	C_{36}	216
Hydrogen,	4·09	3·89	H_{21}	21
Nitrogen,	...	2·59	N	14
Oxygen,	...	8·91	O_6	48
Chlorine,	...	26·31	Cl_4	142
Platinum,	18·29	18·28	Pt	98·7
	100·00			539·7

VII. *Action of Cyanogen on Codeine.*

Dicyanocodeine.—When a current of cyanogen is passed into a solution of codeine in the smallest possible quantity of alcohol, the gas is rapidly absorbed, and the fluid acquires, first a yellow, and, by continued action, a brown colour. If the solution be then left to itself for some time, the smell of cyanogen disappears, and is replaced by that of hydrocyanic acid, and crystals are gradually deposited. In order to obtain the new compound in sufficient quantity, it is best to keep up a continuous slow current of cyanogen, by which means crystals are deposited during the action in considerable abundance. These are collected on a filter, and washed with a small quantity of alcohol; and the filtrate, on being again exposed to the action of cyanogen, yields an additional quantity of crystals inferior in purity to those obtained in the first part of the operation. The product is purified by solution with the aid of heat, in a mixture of alcohol and ether, from which it is deposited in crystals, which are colourless, or slightly yellow. Obtained in this way, however, they are apt to retain a small quantity of codeine: and it is, therefore, advantageous to pass cyanogen into the mixture to be used for their solution, by which means the last traces of codeine are converted into the new compound.

The substance so obtained is a new base, to which I give the name of Dicyanocodeine. It is soluble in boiling absolute alcohol, or a mixture of alcohol and ether, and is deposited on cooling in thin six-sided plates, with a brilliant lustre. It is difficultly soluble in water, but on the addition of alcohol it is dissolved; nothing, however, is deposited from the solution on standing, and by evaporation it is decomposed, and crystals of codeine are left behind. With hydrochloric acid, it is converted into a crystalline salt, but decomposition takes place immediately; for on the addition of potash to the fluid, ammonia escapes, and if it be left for four-and-twenty hours, hydrocyanic acid is evolved. With sulphuric and oxalic acid, it likewise gives somewhat sparingly soluble compounds, which decompose rapidly with the evolution of ammonia and hydrocyanic acid. The crystals deposited from alcohol and ether are anhydrous. Their analysis gave the following results :—

I. { 4·552 grains, dried in vacuo, gave
11·388 ... of carbonic acid, and
2·431 ... of water.

II. { 4·325 grains, dried in vacuo, gave
10·790 ... of carbonic acid, and
2·405 ... of water.

4·954 grains gave by Warrentrap and Will's method 9·320 grains of platinochloride of ammonium.

5·310 grains gave by the same method 9·890 grains of platinochloride.

	Experiment.		Calculation.		
Carbon, . .	68·22	68·04	68·37	C_{40}	240
Hydrogen, . .	5·93	6·17	5·97	H_{21}	21
Nitrogen, . .	11·81	11·50	11·68	N_3	42
Oxygen, . .	14·04	14·27	13·97	O_6	48
	100·00	100·00	100·00		351

These results correspond exactly with the formula $C_{40}H_{21}N_3O_6$. The method of its formation, however, indicates unequivocally, that its rational formula must be $C_{36}H_{21}NO_6\,2C_2N$, representing it as formed by two equivalents of cyanogen coupled with one of codeine, and belonging to the same class of compounds as cyaniline. It differs, however, from that substance, in containing two equivalents of cyanogen; and owing to this circumstance, I was at first inclined to take a different view of its constitution, and to consider it as the hydrocyanate of a cyanocodeine formed by substitution, and represented by the formula $C_{36}H_{20}CyNO_6 + HCy$, according to which its formation could obviously be equally well explained, and I considered the evolution of hydrocyanic acid, by treating it with acids, as favourable to this view. Attentive observation, however, convinced me, that though hydrocyanic acid always is produced by heating it with strong acids, it is never evolved immediately, as it necessarily must be, if it existed as such; but that it only makes its appearance after the lapse of some time, and that only as the result of an advanced decomposition; for long before it is observed, the addition of potash to the acid solution causes an abundant evolution of ammonia.

The ease with which dicyanocodeine is decomposed has prevented my examining any of its compounds. I attempted to prepare a platinum salt by rapid solution in hydrochloric acid, and precipitation by bichloride of platinum; but the instant the latter substance was added, evolution of hydrocyanic acid was observed, and the results obtained were, as might be expected, wholly incongruous and unsatisfactory. The decompositions of dicyanocodeine evidently afford several different substances; but I have not attempted to follow them out, as their investigation seemed to present some difficulties, among which, not the least was that of obtaining the base itself in sufficient quantity.

VIII. *Action of Alkalies on Codeine.*

Codeine, when treated at moderate temperatures with potash, yields more than one volatile base, according to the circumstances in which the experiment is made. I have found that similar results are obtained by the use of hydrate of potash, or of potash-lime, or soda-lime prepared in the usual way. The method employed in the experiment was to mix codeine with four or five times its weight of potash-lime or soda-lime, and introduce the mixture into a retort with a tubulated receiver, having a doubly-bent tube attached to its tubulature, the end of which passed into a small flask containing hydrochloric acid, in order to retain any of the very volatile base which might not be condensed in the receiver. The retort was introduced into an oil-bath, and kept at a uniform temperature of 250° Fahr. As soon as this temperature is reached, a slight peculiar odour is observed, which soon becomes more powerful, and a small quantity of water, retaining the bases in solution, collects in the receiver. The decomposition at 250°, however, is excessively slow, and even after many days, bases are evolved apparently in undiminished quantity, but I retained the mixture steadily at this point, in hopes of obtaining the product free from ammonia, which my preliminary trials had shewn to be produced at higher temperatures; but I found that even with this low heat it was evolved always in appreciable, and, in some experiments, even in considerable quantity. I therefore gradually raised the temperature to about 350°, when a larger quantity of base was obtained; and after the heat had been sustained for some time, small crystals made their appearance, which deposited themselves in a line round the retort, just above the level of the oil in the bath, but which soon rose into and collected in the neck of the retort.

These crystals resemble benzoic acid in their external appearance, and are at first perfectly colourless, but soon acquire a brownish shade by exposure to light and air. They are a base, and rapidly restore the colour of reddened litmus. They are sparingly soluble in water, but readily in acids, and give a precipitate with bichloride of platinum. The quantity of this substance obtained was excessively minute; and though considerable quantities of codeine were operated upon, all that was obtained served only to make the few qualitative experiments now detailed.

The watery fluid which collected in the receiver possessed a pungent and peculiar smell; it restored the colour of reddened litmus with great rapidity, and gave abundant fumes with hydrochloric acid. On the addition of solid potash, a highly volatile and pungent oily base collected as a layer on the surface of the fluid, and at the same time a gaseous base escaped along with ammonia. From the small quantity of these substances which I was able to obtain, I could not attempt to prepare either of them in a pure state. I was therefore under the necessity of determining their constitution by the analysis of their platinum salts, which can be separated from one another, though not without difficulty. In order to prepare these salts, the basic fluid was saturated with hydrochloric acid, and evaporated to dryness in the water-bath, when it left behind a beautifully crystalline mass, highly soluble in water, and deliquescent in moist air. This was dissolved in absolute alcohol, to separate ammonia, and the filtered solution mixed with an alcoholic solution of bichloride of platinum, when the platinum salts were immediately thrown down as a pale-yellow powder, very sparingly soluble in absolute alcohol, but readily dissolved on the addition of water. The separation of the two bases is best effected by heating the washed precipitate with boiling absolute alcohol, and adding water in small quantities until the whole is dissolved. The crystals which deposit on cooling are one of the salts in a state of purity, if the process have been properly managed, or, at all events, only require a repetition of the process to make them absolutely pure. The salt thus obtained is scarcely soluble in absolute alcohol or ether, but is readily soluble in water and dilute spirit, and is thrown down from the latter solution by ether in the form of fine yellow scales. Its analysis gave the following results:—

8·723 grains, dried at 212°, gave
1·753 ... of carbonic acid, and
2·090 ... of water.

9·880 grains, dried at 212°, gave 4·100 grains of platinum.
7·734 ... ... 3·196 ... ...

	Experiment.		Calculation.		
	I.	II.			
Carbon, . . .	5·48	...	5·06	C_2	12
Hydrogen, . .	2·66	...	2·52	H_6	6
Nitrogen, . .	...	...	5·90	N	14
Chlorine, . .	...	...	44·91	Cl_3	106·5
Platinum, . .	41·49	41·32	41·61	Pt	98·7
			100·00		237·2

The formula of the salt is therefore $C_2 H_5 N H Cl Pt Cl_2$; and the base is, consequently, the methylamine of Wurtz, with whose description of that substance and its platinum salt it perfectly agrees.

The preparation of the platinum salt of the other base was attended with much greater difficulty; and I did not succeed in obtaining it quite free from methylamine. In order to obtain it, the fluid which had deposited the methylamine salt was evaporated to a small bulk, the salt which separated filtered off, and ether added to the mother-liquor. Immediately a precipitate is obtained, generally in the form of minute yellow-needles, but sometimes in scales. It is sparingly soluble in alcohol and ether, and highly soluble in water, from which it crystallises in long needles, and with such facility, that a few drops evaporated on a watch-glass leave the salt they contain in the form of five or six needles crossing the whole space occupied by the solution. The quantity of this salt which I had at my disposal was too small to admit of my carrying its purification by recrystallisation as far as was to be desired, and, consequently, a small quantity of methylamine remained in those subjected to analysis.

I. 5·521 grains of platinum salt, dried at 212°, gave
2·485 ... of carbonic acid, and
1·800 ... of water.

10·475 grains platinum salt, gave 3·951 grains platinum.
6·475 ... ... 2·432 ... ...

	Experiment.		Calculation.		
	I.	II.			
Carbon, . . .	12·27	...	13·57	C_6	36
Hydrogen, . .	3·62	...	3·77	H_{10}	10
Nitrogen, . .	...	...	5·27	N	14
Chlorine, . .	...	...	40·18	Cl_3	106·5
Platinum, . .	37·71	37·56	37·21	Pt	98·7
			100·00		265·2

These results approach most closely to the formula $C_6 H_9 N H Cl Pt Cl_2$; and though the carbon is very deficient, and the platinum considerably in excess, there can be no doubt that this is due to the imperfect separation of the methylamine, and that this is its true formula; and that of the base itself $C_6 H_9 N$. The base, then, obviously belongs to the same series as methylamine, and forms the term of the series corresponding to metacetonic acid, and, in accordance with the system of nomenclature adopted by WURTZ, it receives the name of metacetamine. I have not attempted the examination of the salts of this base, as I did not obtain it in sufficient quantity for that purpose; but I take the opportunity of stating, that before I had obtained it from codeine I had ascertained its existence among the products of destructive distillation of animal substances, and that I shall, at a future period, detail the properties of its compounds.*

The residue in the retort after these bases have been evolved, is dark cinnamon-brown, and slightly coherent; it dissolves in water, with a dark-brown, almost black colour, and gives with acids a flocculent brown precipitate of a humus-like substance, and perfectly amorphous, which I have not thought it necessary to examine. It still contains nitrogen; and by exposure to a heat gradually raised to low redness, it gives an additional quantity of volatile bases, among which ammonia becomes more and more abundant as the temperature rises. A non-basic oil also makes its appearance, but only in very small quantity.

Since these experiments were made, I have received the February number of the *Annalen der Chimie und Pharmacie*, which contains a preliminary notice of an investigation by WERTHEIM of the action of soda-lime on certain organic bases. He has obtained metacetamine from narcotine, and methylamine from morphia; and considering these substances to be directly eliminated from the bases, he expects to obtain the residual atoms in the form of a definite compound. I entertained a similar idea with regard to codeine, until I detected the formation of two different bases, which seemed to me rather to indicate that these substances appear as the result of a true destructive distillation; and that possibly by varying the circumstances of the experiment, other bases may be obtained.

I have also observed another remarkable decomposition of codeine, by which volatile bases are obtained. I have already mentioned the formation, by the action of nitric acid, of a resinous acid, with the examination of which I am still engaged. This acid, which is insoluble in water, dissolves readily in dilute potash, with a red colour; and the solution on boiling evolves a volatile base in great abundance. I have not yet determined the whole circumstances under which this change takes place, but reserve this for a future communication.*

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* I may at the same time mention, that I have convinced myself that the petinine described by me two years since as existing in bone-oil, is represented by the formula $C_8 H_{11} N$, and not by $C_8 H_{10} N$, which I then gave for it. Indeed, my analysis of the platinum salt, which is most to be depended upon, tallies equally well with either formula. I have also ascertained the existence of ethylamine and methylamine in bone-oil. The details of these experiments will be contained in the second part of my paper on the Products of the Destructive Distillation of Animal Matters.

I have likewise examined the action of iodine on codeine, which yields a magnificent crystalline compound presenting the phenomena of pleochroism in a remarkable manner. Difficulties connected with the analysis have, however, prevented my hitherto completing its investigation.

The following is a Tabular View of the constitutions of the substances described in this paper:—

Codeine,	$C_{36} H_{21} NO_6$.
... crystallised,	$C_{36} H_{21} NO_6 + 2 HO$.
Hydrochlorate,	$C_{36} H_{21} NO_6 HCl + 4 HO$.
Hydriodate,	$C_{36} H_{21} NO_6 HI + 2 HO$.
Sulphate,	$C_{36} H_{21} NO_6 HO SO_3 + 5 HO$.
Nitrate,	$C_{36} H_{21} NO_6 HO NO_5$.
Phosphate,	$(C_{36} H_{21} NO_6 HO) 2 HO PO_5 + 3 HO$.
Oxalate,	$C_{36} H_{21} NO_6 HO C_2 O_3 + 3 HO$.
Hydrosulphocyanate,	$C_{36} H_{21} NO_6 HC_2 NS_2 + HO$.
Platinum salt dried at 212°,	$C_{36} H_{21} NO_6 HCl Pt Cl_2 + HO$.
... crystallised,	$C_{36} H_{21} NO_6 HCl Pt Cl_2 + 3 HO$.
Amorphous codeine,	$C_{36} H_{21} NO_6$.
Nitrocodeine,	$C_{36} H_{20} (NO_4) NO_6$.
Sulphate,	$C_{36} H_{20} (NO_4) NO_6 HO SO_3$.
Platinum salt,	$C_{36} H_{20} (NO_4) NO_6 HCl Pt Cl_2 + 4 HO$.
Bromocodeine,	$C_{36} H_{20} Br NO_6$.
... hydrate,	$C_{36} H_{20} Br NO_6 + HO$.
... terhydrate,	$C_{36} H_{20} Br NO_6 + 3 HO$.
Hydrobromate,	$C_{36} H_{20} Br NO_6 HBr + 2 HO$.
Platinum salt,	$C_{36} H_{20} Br NO_6 HCl Pt Cl_2$.
Tribromocodeine,	$C_{36} H_{18} Br_3 NO_6$.
Hydrobromate,	$2 (C_{36} H_{18} Br_3 NO_6) 3 HBr$.
Platinum salt,	$C_{36} H_{18} Br_3 NO_6 HCl Pt Cl_2$.
Chlorocodeine,	$C_{36} H_{20} Cl NO_6$.
... terhydrate,	$C_{36} H_{20} Cl NO_6 + 3 HO$.
Sulphate,	$C_{36} H_{20} Cl NO_6 HO SO_3 + 4 HO$.
Platinum salt,	$C_{36} H_{20} Cl NO_6 HCl Pt Cl_2$.
Dicyanocodeine,	$C_{36} H_{21} NO_6 2 C_2 N$.
Metacetamine,	$C_6 H_9 N$.

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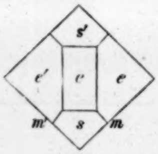
* The action of nitric acid on the organic alkalies, in this point of view, is now under investigation in my laboratory. Narcotine has been found to undergo a precisely similar change, yielding a compound, which gives off a volatile base by ebullition with potash, and a whole series of other substances, the constitution of which will be detailed so soon as the investigations are completed.

SUPPLEMENT.

While engaged with the investigation of codeine, I sent to Professor Miller, of Cambridge, some crystals of the base and its sulphate for crystallographic measurement. Owing to Professor Miller's other avocations, he was unable to furnish me with the results in sufficient time to admit of their being incorporated with the foregoing paper. I have, therefore, introduced them here in the shape of a supplement, as they form a valuable addition to the observations contained in the paper.

Codeine.—Prismatic. The symbols of the simple forms are, *c* 001, *s* 011, *e* 101, *u* 102, *m* 110. The angles between normals to the faces are:

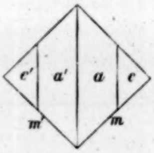
Fig. 1.



	°	′
m c	90	0
s c	38	37
s s'	77	14
e c	39	46
e e'	79	32
u c	22	35
u u'	45	10
m m'	87	40
e m	63	42
s m	63	15
s e	53	3

Cleavage, *c*.

Fig. 2.



The faces *m m'* are usually of very unequal magnitude. The faces *s s'* were not observed upon the same crystals. The form *s* is probably hemihedral.

Fig. 1. Codeine crystallised from alcohol.

Fig. 2. Codeine crystallised from water.

The agreement of the different observations is not very good, so that the above measures must be considered as approximations only.

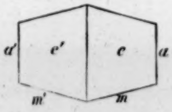
Sulphate of Codeine.—Prismatic. Symbols of the simple forms, *a* 100, *e* 101, *m* 110.

The angles between normals to the faces are:

	°	′
e a	66	15
e e'	46	30
m a	75	36
m m'	28	48

Cleavage, *a*.

Fig. 3.



IV.—*On the Equilibrium of Elastic Solids.* By JAMES CLERK MAXWELL, Esq.

(Read 18th February, 1850).

There are few parts of mechanics in which theory has differed more from experiment than in the theory of elastic solids.

Mathematicians, setting out from very plausible assumptions with respect to the constitution of bodies, and the laws of molecular action, came to conclusions which were shewn to be erroneous by the observations of experimental philosophers. The experiments of ŒRSTED proved to be at variance with the mathematical theories of NAVIER, POISSON, and LAMÉ and CLAPEYRON, and apparently deprived this practically important branch of mechanics of all assistance from mathematics.

The assumption on which these theories were founded may be stated thus:—

Solid bodies are composed of distinct molecules, which are kept at a certain distance from each other by the opposing principles of attraction and heat. When the distance between two molecules is changed, they act on each other with a force whose direction is in the line joining the centres of the molecules, and whose magnitude is equal to the change of distance multiplied into a function of the distance which vanishes when that distance becomes sensible.

The equations of elasticity deduced from this assumption contain only *one* coefficient, which varies with the nature of the substance.

The insufficiency of one coefficient may be proved from the existence of bodies of different degrees of solidity.

No effort is required to retain a liquid in any form, if its volume remain unchanged; but when the form of a solid is changed, a force is called into action which tends to restore its former figure; and this constitutes the difference between elastic solids and fluids. Both tend to recover their *volume,* but fluids do not tend to recover their *shape.*

Now, since there are in nature bodies which are in every intermediate state from perfect solidity to perfect liquidity, these two elastic powers cannot exist in every body in the same proportion, and therefore all theories which assign to them an invariable ratio must be erroneous.

I have therefore substituted for the assumption of NAVIER the following axioms as the results of experiments.

If three pressures in three rectangular axes be applied at a point in an elastic solid,—

1. *The sum of the three pressures is proportional to the sum of the compressions which they produce.*

2. *The difference between two of the pressures is proportional to the difference of the compressions which they produce.*

The equations deduced from these axioms contain two coefficients, and differ from those of NAVIER only in not assuming any invariable ratio between the cubical and linear elasticity. They are the same as those obtained by Professor STOKES from his equations of fluid motion, and they agree with all the laws of elasticity which have been deduced from experiments.

In this paper *pressures* are expressed by the number of units of weight to the unit of surface; if in English measure, in pounds to the square inch, or in atmospheres of 15 pounds to the square inch.

Compression is the proportional change of any dimension of the solid caused by pressure, and is expressed by the quotient of the change of dimension divided by the dimension compressed.*

Pressure will be understood to include tension, and compression dilatation; pressure and compression being reckoned positive.

Elasticity is the force which opposes pressure, and the *equations of elasticity* are those which express the relation of pressure to compression.†

Of those who have treated of elastic solids, some have confined themselves to the investigation of the laws of the bending and twisting of rods, without considering the relation of the coefficients which occur in these two cases; while others have treated of the general problem of a solid body exposed to any forces.

The investigations of LEIBNITZ, BERNOULLI, EULER, VARIGNON, YOUNG, LA HIRE, and LAGRANGE, are confined to the equilibrium of bent rods; but those of NAVIER, POISSON, LAMÉ and CLAPEYRON, CAUCHY, STOKES, and WERTHEIM, are principally directed to the formation and application of the general equations.

The investigations of NAVIER are contained in the seventh volume of the *Memoirs of the Institute*, page 373; and in the *Annales de Chimie et de Physique*, 2^e *Série*, xv., 264, and xxxviii., 435; *L'Application de la Mécanique*, tom. i.

Those of POISSON in *Mém. de l'Institut*, viii., 429; *Annales de Chimie*, 2^e *Série*, xxxvi., 334; xxxvii., 337; xxxviii., 338; xlii. *Journal de l'École Polytechnique*, cahier xx., with an abstract in *Annales de Chimie* for 1829.

The memoir of MM. LAMÉ and CLAPEYRON is contained in CRELLE's *Mathematical Journal*, vol. vii.; and some observations on elasticity are to be found in LAMÉ's *Cours de Physique*.

M. CAUCHY's investigations are contained in his *Exercises de Analyse*, vol. iii., p. 180, published in 1828.

Instead of supposing each pressure proportional to the linear compression which it produces, he supposes it to consist of two parts, one of which is proportional to the linear compression in the direction of the pressure, while the other is proportional to the diminution of volume. As this hypothesis admits two coefficients, it differs from that of this paper only in the values of the coefficients selected. They are denoted by K and k, and $K=\mu-\frac{1}{3}m$, $k=m$.

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* The laws of pressure and compression may be found in the Memoir of Lamé and Clapeyron. See note A.

† See note B.

The theory of Professor STOKES is contained in Vol. viii., Part 3, of the *Cambridge Philosophical Transactions*, and was read April 14, 1845.

He states his general principles thus:—"The capability which solids possess of being put into a state of isochronous vibration, shews that the pressures called into action by small displacements depend on homogeneous functions of those displacements of one dimension. I shall suppose, moreover, according to the general principle of the superposition of small quantities, that the pressures due to different displacements are superimposed, and, consequently, that the pressures are linear functions of the displacements."

Having assumed the proportionality of pressure to compression, he proceeds to define his coefficients.—"Let $-A\delta$ be the pressures corresponding to a uniform linear dilatation δ when the solid is in equilibrium, and suppose that it becomes $mA\delta$, in consequence of the heat developed when the solid is in a state of rapid vibration. Suppose, also, that a displacement of shifting parallel to the plane xy, for which $\delta x=kx$, $\delta y=-ky$, and $\delta z=0$, calls into action a pressure $-Bk$ on a plane perpendicular to the axis of x, and a pressure Bk on a plane perpendicular to the axis of y; the pressure on these planes being equal and of contrary signs; that on a plane perpendicular to z being zero, and the tangential forces on those planes being zero." The coefficients A and B, thus defined, when expressed as in this paper, are $A=3\mu$, $B=\frac{m}{2}$.

Professor STOKES does not enter into the solution of his equations, but gives their results in some particular cases.

1. A body exposed to a uniform pressure on its whole surface.
2. A rod extended in the direction of its length.
3. A cylinder twisted by a statical couple.

He then points out the method of finding A and B from the two last cases.

While explaining why the equations of motion of the luminiferous ether are the same as those of incompressible elastic solids, he has mentioned the property of *plasticity* or the tendency which a constrained body has to relieve itself from a state of constraint, by its molecules assuming new positions of equilibrium. This property is opposed to linear elasticity; and these two properties exist in all bodies, but in variable ratio.

M. WERTHEIM, in *Annales de Chimie*, 3^e *Série*, xxiii., has given the results of some experiments on caoutchouc, from which he finds that $K=k$, or $\mu=\frac{4}{3}m$; and concludes that $k=K$ in all substances. In his equations, μ is therefore made equal to $\frac{4}{3}m$.

The accounts of experimental researches on the values of the coefficients are so numerous that I can mention only a few.

Canton, Perkins, Œrsted, Aimé, Colladon and Sturm, and Regnault, have determined the cubical compressibilities of substances; Coulomb, Duleau, and Giulio, have calculated the linear elasticity from the torsion of wires; and a great many observations have been made on the elongation and bending of beams.

I have found no account of any experiments on the relation between the doubly refracting power communicated to glass and other elastic solids by compression, and the pressure which produces it;* but the phenomena of bent glass seem to prove, that, in homogeneous singly-refracting substances exposed to pressures, the principal axes of pressure coincide with the principal axes of double refraction; and that the difference of pressures in any two axes is proportional to the difference of the velocities of the oppositely polarised rays whose directions are parallel to the third axis. On this principle I have calculated the phenomena seen by polarised light in the cases where the solid is bounded by parallel planes.

In the following pages I have endeavoured to apply a theory identical with that of Stokes to the solution of problems which have been selected on account of the possibility of fulfilling the conditions. I have not attempted to extend the theory to the case of imperfectly elastic bodies, or to the laws of permanent bending and breaking. The solids here considered are supposed not to be compressed beyond the limits of perfect elasticity.

The equations employed in the transformation of co-ordinates may be found in Gregory's *Solid Geometry.*

I have denoted the displacements by $\delta x, \delta y, \delta z$. They are generally denoted by α, β, γ; but as I had employed these letters to denote the principal axes at any point, and as this had been done throughout the paper, I did not alter a notation which to me appears natural and intelligible.

The laws of elasticity express the relation between the changes of the dimensions of a body and the forces which produce them.

These forces are called Pressures, and their effects Compressions. Pressures are estimated in pounds on the square inch, and compressions in fractions of the dimensions compressed.

Let the position of material points in space be expressed by their co-ordinates x, y, and z, then any change in a system of such points is expressed by giving to these co-ordinates the variations $\delta x, \delta y, \delta z$, these variations being functions of x, y, z.

The quantities $\delta x, \delta y, \delta z$, represent the absolute motion of each point in the directions of the three co-ordinates; but as compression depends not on absolute, but on relative displacement, we have to consider only the nine quantities—

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* See note C.

$$\begin{array}{ccc} \dfrac{d\,\delta x}{dx} & \dfrac{d\,\delta x}{dy} & \dfrac{d\,\delta x}{dz} \\[2ex] \dfrac{d\,\delta y}{dx} & \dfrac{d\,\delta y}{dy} & \dfrac{d\,\delta y}{dz} \\[2ex] \dfrac{d\,\delta z}{dx} & \dfrac{d\,\delta z}{dy} & \dfrac{d\,\delta z}{dz} \end{array}$$

Since the number of these quantities is nine, if nine other independent quantities of the same kind can be found, the one set may be found in terms of the other. The quantities which we shall assume for this purpose are—

1. Three compressions, $\frac{\delta\alpha}{\alpha}$, $\frac{\delta\beta}{\beta}$, $\frac{\delta\gamma}{\gamma}$, in the directions of three principal axes α, β, γ.

2. The nine *direction-cosines* of these axes, with the *six connecting equations*, leaving three independent quantities. (See GREGORY'S *Solid Geometry*).

3. The small angles of rotation of this system of axes about the axes of x, y, z,

The cosines of the angles which the axes of x, y, z make with those of α, β, γ are—

$$\cos(\alpha\, 0\, x)=a_1,\ \cos(\beta\, 0\, x)=b_1,\ \cos(\gamma\, 0\, x)=c_1,$$
$$\cos(\alpha\, 0\, y)=a_2,\ \cos(\beta\, 0\, y)=b_2,\ \cos(\gamma\, 0\, y)=c_2,$$
$$\cos(\alpha\, 0\, z)=a_3,\ \cos(\beta\, 0\, z)=b_3,\ \cos(\gamma\, 0\, z)=c_3,$$

These *direction-cosines* are connected by the six equations,

$$\begin{array}{ll} a_1^2+b_1^2+c_1^2=1 & a_1 a_2+b_1 b_2+c_1 c_2=0 \\ a_2^2+b_2^2+c_2^2=1 & a_2 a_3+b_2 b_3+c_2 c_3=0 \\ a_3^2+b_3^2+c_3^2=1 & a_3 a_1+b_3 b_1+c_3 c_1=0 \end{array}$$

The rotation of the system of axes α, β, γ, round the axis of

$$x,\text{ from } y \text{ to } z,\ =\delta\,\theta_1,$$
$$y,\text{ from } z \text{ to } x,\ =\delta\,\theta_2,$$
$$z,\text{ from } x \text{ to } y,\ =\delta\,\theta_3;$$

By resolving the displacements $\delta\alpha$, $\delta\beta$, $\delta\gamma$, θ_1, θ_2, θ_3, in the directions of the axes x, y, z, the displacements in these axes are found to be

$$\delta x = a_1\,\delta\alpha + b_1\,\delta\beta + c_1\,\delta\gamma - \theta_2 z + \theta_3 y$$
$$\delta y = a_2\,\delta\alpha + b_2\,\delta\beta + c_2\,\delta\gamma - \theta_3 x + \theta_1 z$$
$$\delta z = a_3\,\delta\alpha + b_3\,\delta\beta + c_2\,\delta\gamma - \theta_1 y + \theta_2 x$$

But $$\delta\alpha = \alpha\frac{\delta\alpha}{\alpha},\ \delta\beta = \beta\frac{\delta\beta}{\beta},\text{ and } \delta\gamma = \gamma\frac{\delta\gamma}{\gamma},$$

and $$\alpha=a_1 x+a_2 y+a_3 z,\ \beta=b_1 x+b_2 y+b_3 z,\text{ and } \gamma=c_1 x+c_2 y+c_3 z.$$

Substituting these values of $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ in the expressions for δx, δy,

δz, and differentiating with respect to x, y, and z, in each equation, we obtain the equations—

$$\left.\begin{aligned}
\frac{d\,\delta x}{d x} &= \frac{\delta \alpha}{\alpha} a_1^2 + \frac{\delta \beta}{\beta} b_1^2 + \frac{\delta \gamma}{\gamma} c_1^2 \\
\frac{d\,\delta y}{d y} &= \frac{\delta \alpha}{\alpha} a_2^2 + \frac{\delta \beta}{\beta} b_2^2 + \frac{\delta \gamma}{\gamma} c_2^2 \\
\frac{d\,\delta z}{d z} &= \frac{\delta \alpha}{\alpha} a_3^2 + \frac{\delta \beta}{\beta} b_3^2 + \frac{\delta \gamma}{\gamma} c_3^2
\end{aligned}\right\} \quad (1).$$

Equations of compression.

$$\left.\begin{aligned}
\frac{d\,\delta x}{d y} &= \frac{\delta \alpha}{\alpha} a_1 a_2 + \frac{\delta \beta}{\beta} b_1 b_2 + \frac{\delta \gamma}{\gamma} c_1 c_2 + \delta \theta_3 \\
\frac{d\,\delta x}{d z} &= \frac{\delta \alpha}{\alpha} a_1 a_3 + \frac{\delta \beta}{\beta} b_1 b_3 + \frac{\delta \gamma}{\gamma} c_1 c_3 - \delta \theta_2 \\
\frac{d\,\delta y}{d z} &= \frac{\delta \alpha}{\alpha} a_2 a_3 + \frac{\delta \beta}{\beta} b_2 b_3 + \frac{\delta \gamma}{\gamma} c_2 c_3 + \delta \theta_1 \\
\frac{d\,\delta y}{d x} &= \frac{\delta \alpha}{\alpha} a_2 a_1 + \frac{\delta \beta}{\beta} b_2 b_1 + \frac{\delta \gamma}{\gamma} c_2 c_1 - \delta \theta_3 \\
\frac{d\,\delta z}{d x} &= \frac{\delta \alpha}{\alpha} a_3 a_1 + \frac{\delta \beta}{\beta} b_3 b_1 + \frac{\delta \gamma}{\gamma} c_3 c_1 + \delta \theta_2 \\
\frac{d\,\delta z}{d y} &= \frac{\delta \alpha}{\alpha} a_3 a_2 + \frac{\delta \beta}{\beta} b_3 b_2 + \frac{\delta \gamma}{\gamma} c_3 c_2 - \delta \theta_1
\end{aligned}\right\} \quad (2).$$

Equations of the equilibrium of an element of the solid.

The forces which may act on a particle of the solid are:—

1. Three attractions in the direction of the axes, represented by X, Y, Z.
2. Six pressures on the six faces.
3. Two tangential actions on each face.

Let the six faces of the small parallelopiped be denoted by x_1, y_1, z_1, x_2, y_2, and z_2, then the forces acting on x_1 are:—

1. A normal pressure p_1 acting in the direction of x on the area $d y\, d z$.
2. A tangential force q_3 acting in the direction of y on the same area.
3. A tangential force q_2^1 acting in the direction of z on the same area, and so on for the other five faces, thus:—

Forces which act in the direction of the axes of

	x	y	z
On the face x_1	$-p_1\, d y\, d z$	$-q_3\, d y\, d z$	$-q_2^1\, d y\, d z$
… … x_2	$(p_1 + \frac{d p_1}{d x} d x)\, d y\, d z$	$(q_3 + \frac{d q_3}{d x} d x)\, d y\, d x$	$(q_2^1 + \frac{d q_2^1}{d x} d x)\, d y\, d z$
… … y_1	$-q_3^1\, d z\, d x$	$-p_2\, d z\, d x$	$-q_1\, d z\, d x$
… … y_2	$(q_3 + \frac{d q_3^1}{d y} d y)\, d z\, d x$	$(p_2 + \frac{d p_2}{d y} d y)\, d z\, d x$	$(q_1 + \frac{d q_1}{d y} d y)\, d z\, d x$

On the face	z_1	$-q_2\,dx\,dy$	$-q^1\,dx\,dy$	$-p_3\,dx\,dy$
... ...	z_2	$(q_2+\frac{dq_2}{dz}\,dz)\,dx\,dy$	$(q^1+\frac{dp^1}{dz}\,dz)\,dx\,dy$	$(p_3+\frac{dp_3}{dz}\,dz)\,dx\,dy$
Attractions,		$\rho\,\mathrm{X}\,dx\,dy\,dz$	$\rho\,\mathrm{Y}\,dx\,dy\,dz$	$\rho\,\mathrm{Z}\,dx\,dy\,dz$

Taking the moments of these forces round the axes of the particle, we find

$$q_1{}^1=q_1 \qquad q_1{}^2=q_2 \qquad q_3{}^1=q_3\,;$$

and then equating the forces in the directions of the three axes, and dividing by dx, dy, dz, we find the equations of pressures.

$$\left.\begin{aligned}\frac{dp_1}{dx}+\frac{dq_3}{dy}+\frac{dq_2}{dz}+\rho\,\mathrm{X}&=0\\ \frac{dp_2}{dy}+\frac{dq_1}{dz}+\frac{dq_3}{dx}+\rho\,\mathrm{Y}&=0\\ \frac{dp_3}{dz}+\frac{dq_2}{dx}+\frac{dq_1}{dy}+\rho\,\mathrm{Z}&=0\end{aligned}\right\}\ \text{Equations of Pressures.}\ \ldots\ (3.)$$

The resistance which the solid opposes to these pressures is called Elasticity, and is of two kinds, for it opposes either change of *volume* or change of *figure*. These two kinds of elasticity have no necessary connection, for they are possessed in very different ratios by different substances. Thus *jelly* has a cubical elasticity little different from that of water, and a linear elasticity as small as we please; while *cork*, whose cubical elasticity is very small, has a much greater linear elasticity than jelly.

HOOKE discovered that the elastic forces are proportional to the changes that excite them, or, as he expressed it, "Ut tensio sic vis."

To fix our ideas, let us suppose the compressed body to be a parallelopiped, and let pressures P_1, P_2, P_3 act on its faces in the direction of the axes α, β, γ, which will become the principal axes of compression, and the compressions will be $\frac{\delta\alpha}{\alpha}, \frac{\delta\beta}{\beta}, \frac{\delta\gamma}{\gamma}$.

The fundamental assumption from which the following equations are deduced is an extension of HOOKE's law, and consists of two parts.

I. The sum of the compressions is proportional to the sum of the pressures.

II. The difference of the compressions is proportional to the difference of the pressures.

These laws are expressed by the following equations :—

$$\text{I.}\ (\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3)=3\mu\left(\frac{\delta\alpha}{\alpha}+\frac{\delta\beta}{\beta}+\frac{\delta\gamma}{\gamma}\right)\ .\ (4.)$$

Equations of Elasticity.

$$\text{II.}\ \left\{\begin{aligned}(\mathrm{P}_1-\mathrm{P}_2)&=m\left(\frac{\delta\alpha}{\alpha}-\frac{\delta\beta}{\beta}\right)\\ (\mathrm{P}_2-\mathrm{P}_3)&=m\left(\frac{\delta\beta}{\beta}-\frac{\delta\gamma}{\gamma}\right)\\ (\mathrm{P}_3-\mathrm{P}_1)&=m\left(\frac{\delta\gamma}{\gamma}-\frac{\delta\alpha}{\alpha}\right)\end{aligned}\right\}\ .\ (5.)$$

The quantity μ is the coefficient of cubical elasticity, and m that of linear elasticity.

By solving these equations, the values of the pressures P_1, P_2, P_3, and the compressions $\frac{\delta a}{a}$, $\frac{\delta \beta}{\beta}$, $\frac{\delta \gamma}{\gamma}$ may be found.

$$\left.\begin{aligned}
P_1 &= (\mu - \tfrac{1}{3} m)\left(\frac{\delta a}{a} + \frac{\delta \beta}{\beta} + \frac{\delta \gamma}{\gamma}\right) + m \frac{\delta a}{a} \\
P_2 &= (\mu - \tfrac{1}{3} m)\left(\frac{\delta a}{a} + \frac{\delta \beta}{\beta} + \frac{\delta \gamma}{\gamma}\right) + m \frac{\delta \beta}{\beta} \\
P_3 &= (\mu - \tfrac{1}{3} m)\left(\frac{\delta a}{a} + \frac{\delta \beta}{\beta} + \frac{\delta \gamma}{\gamma}\right) + m \frac{\delta \gamma}{\gamma}
\end{aligned}\right\} \quad . \quad (6.)$$

$$\left.\begin{aligned}
\frac{\delta a}{a} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(P_1 + P_2 + P_3) + \frac{1}{m} P_1 \\
\frac{\delta \beta}{\beta} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(P_1 + P_2 + P_3) + \frac{1}{m} P_2 \\
\frac{\delta \gamma}{\gamma} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right) P_1 + P_2 + P_3) + \frac{1}{m} P_3
\end{aligned}\right\} \quad . \quad (7.)$$

From these values of the pressures in the axes a, β, γ, may be obtained the equations for the axes x, y, z, by resolution of pressures and compressions.*

For $\qquad p = a^2 P_1 + b^2 P_2 + c^2 P_3$

and $\qquad q = a a P_1 + b b P_2 + c c P_3$

$$\left.\begin{aligned}
p_1 &= (\mu - \tfrac{1}{3} m)\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz}\right) + m \frac{d\,\delta x}{dx} \\
p_2 &= (\mu - \tfrac{1}{3} m)\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz}\right) + m \frac{d\,\delta y}{dy} \\
p_2 &= (\mu - \tfrac{1}{3} m)\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dx}\right) + m \frac{d\,\delta z}{dz}
\end{aligned}\right\} \quad . \quad (8.)$$

$$\left.\begin{aligned}
q_1 &= \frac{m}{2}\left(\frac{d\,\delta y}{dz} + \frac{d\,\delta z}{dy}\right) \\
q_2 &= \frac{m}{2}\left(\frac{d\,\delta z}{dx} + \frac{d\,\delta x}{dz}\right) \\
q_3 &= \frac{m}{2}\left(\frac{d\,\delta x}{dy} + \frac{d\,\delta y}{dx}\right)
\end{aligned}\right\} \quad . \quad (9.)$$

$$\left.\begin{aligned}
\frac{d\,\delta x}{dx} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + \frac{1}{m} p_1 \\
\frac{d\,\delta y}{dy} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + \frac{1}{m} p_2 \\
\frac{d\,\delta z}{dz} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + \frac{1}{m} p_3
\end{aligned}\right\} \quad . \quad (10.)$$

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* See the Memoir of Lamé and Clapeyron, and note A.

$$\left.\begin{aligned}\frac{d\,\delta x}{dy} - \delta\theta_3 &= \frac{d\,\delta y}{dx} + \delta\theta_3 = \frac{1}{m}q_3 \\ \frac{d\,\delta y}{dz} - \delta\theta_1 &= \frac{d\,\delta z}{dy} + \delta\theta_1 = \frac{1}{m}q_1 \\ \frac{d\,\delta z}{dx} - \delta\theta_2 &= \frac{d\,\delta x}{dz} + \delta\theta_2 = \frac{1}{m}q_2\end{aligned}\right\} \quad . \quad (11.)$$

By substituting in Equations (3.) the values of the forces given in Equations (8.) and (9.), they become

$$\left.\begin{aligned}(\mu + \tfrac{1}{6}m)\left(\frac{d}{dx}\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz}\right)\right) + \frac{m}{2}\left(\frac{d^2}{dx^2}\delta x + \frac{d^2}{dy^2}\delta y + \frac{d^2}{dz^2}\delta z\right) + \rho X = 0 \\ (\mu + \tfrac{1}{6}m)\left(\frac{d}{dy}\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz}\right)\right) + \frac{m}{2}\left(\frac{d^2}{dx^2}\delta x + \frac{d^2}{dy^2}\delta y + \frac{d^2}{dz^2}\delta z\right) + \rho Y = 0 \\ (\mu + \tfrac{1}{6}m)\left(\frac{d}{dz}\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz}\right)\right) + \frac{m}{2}\left(\frac{d^2}{dx^2}\delta x + \frac{d^2}{dy^2}\delta y + \frac{d^2}{dz^2}\delta z\right) + \rho Z = 0\end{aligned}\right\} \quad (12.)$$

These are the general equations of elasticity, and are identical with those of M. Cauchy, in his *Exercises d'Analyse*, vol. iii., p. 180, published in 1828, when k stands for m, and K for $\mu - \frac{m}{2}$, and those of Mr Stokes, given in the *Cambridge Philosophical Transactions*, vol. viii.,[*] part 3, and numbered (30.); in his equations $A = 3\mu$, $B = \frac{m}{2}$.

If the temperature is variable from one part to another of the elastic solid, the compressions $\frac{d\,\delta x}{dx}$, $\frac{d\,\delta y}{dy}$, $\frac{d\,\delta z}{dz}$, at any point will be diminished by a quantity proportional to the temperature at that point. This principle is applied in Cases X. and XI. Equations (10.) then become

$$\left.\begin{aligned}\frac{d\,\delta x}{dx} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + c_3 v + \frac{1}{m}p_1 \\ \frac{d\,\delta y}{dy} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + c_3 v + \frac{1}{m}p_2 \\ \frac{d\,\delta x}{dz} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + c_3 v + \frac{1}{m}p_3\end{aligned}\right\} \quad . \quad . \quad . \quad (13.)$$

$c_3 v$ being the linear expansion for the temperature v.

Having found the general equations of the equilibrium of elastic solids, I proceed to work some examples of their application, which afford the means of determining the coefficients μ, m, and ω, and of calculating the stiffness of solid figures. I begin with those cases in which the elastic solid is a hollow cylinder exposed to given forces on the two concentric cylindric surfaces, and the two parallel terminating planes.

In these cases the co-ordinates x, y, z are replaced by the co-ordinates

$x = X$, measured along the axis of the cylinder.

$y = r$, the radius of any point, or the distance from the axis.

$z = r\theta$, the arc of a circle measured from a fixed plane passing through the axis.

$\frac{d\,\delta x}{dx} = \frac{d\,\delta x}{dx}$, $p_1 = 0$, are the compression and pressure in the direction of the axis at any point.

$\frac{d\,\delta y}{dy} = \frac{d\,\delta r}{dr}$, $p_2 = p$, are the compression and pressure in the direction of the radius.

$\frac{d\,\delta z}{dz} = \frac{d\,\delta r\theta}{d r\theta} = \frac{\delta r}{r}$, $p_3 = q$, are the compression and pressure in the direction of the tangent.

Equations (9.) become, when expressed in terms of these co-ordinates—

$$\left.\begin{aligned} q_1 &= \frac{m}{2}\, r\, \frac{d\,\delta\theta}{dr} \\ q_2 &= \frac{m}{2}\, r\, \frac{d\,\delta\theta}{dx} \\ q_3 &= \frac{m}{2} \cdot \frac{d\,\delta x}{dr} \end{aligned}\right\} \quad . \quad (14.)$$

The length of the cylinder is b, and the two radii a_1 and a_2 in every case.

CASE I.

The first equation is applicable to the case of a hollow cylinder, of which the outer surface is fixed, while the inner surface is made to turn through a small angle $\delta\theta$, by a couple whose moment is M.

The twisting force M is resisted only by the elasticity of the solid, and therefore the whole resistance, in every concentric cylindric surface, must be equal to M.

The resistance at any point, multiplied into the radius at which it acts, is expressed by $r\,q_1 = \frac{m}{2} r^2 \frac{d\,\delta\theta}{dr}$.

Therefore for the whole cylindric surface

$$\frac{d\,\delta\theta}{dr}\, m\, \pi\, r^3\, b = \mathrm{M}.$$

Whence
$$\delta\theta = \frac{\mathrm{M}}{2\,\pi\, m\, b}\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)$$

and
$$m = \frac{\mathrm{M}}{2\,\pi\, b\,\delta\theta}\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right) \quad . \quad . \quad (15.)$$

The optical effect of the pressure of any point is expressed by

$$\mathrm{I} = \omega\, q_1\, b = \omega \frac{\mathrm{M}\, b}{\pi\, r^2} \quad . \quad . \quad (16.)$$

Therefore, if the solid be viewed by polarized light (transmitted parallel to the axis), the difference of retardation of the oppositely polarized rays at any point in the solid will be inversely proportional to the square of the distance from the axis of the cylinder, and the planes of polarization of these rays will be inclined 45° to the radius at that point.

The general appearance is therefore a system of coloured rings arranged oppositely to the rings in uniaxal crystals, the tints ascending in the scale as they approach the centre, and the distance between the rings decreasing towards the centre. The whole system is crossed by two dark bands inclined 45° to the plane of primitive polarization, when the plane of the analysing plate is perpendicular to that of the first polarizing plate.

A jelly of isinglass poured when hot between two concentric cylinders forms, when cold, a convenient solid for this experiment; and the diameters of the rings may be varied at pleasure by changing the force of torsion applied to the interior cylinder.

By continuing the force of torsion while the jelly is allowed to dry, a hard plate of isinglass is obtained, which still acts in the same way on polarized light, even when the force of torsion is removed.

It seems that this action cannot be accounted for by supposing the interior parts kept in a state of constraint by the exterior parts, as in unannealed and heated glass; for the optical properties of the plate of isinglass are such as would indicate a strain preserving in every part of the plate the direction of the original strain, so that the strain on one part of the plate cannot be maintained by an opposite strain on another part.

Two other uncrystallised substances have the power of retaining the polarizing structure developed by compression. The first is a mixture of wax and resin pressed into a thin plate between two plates of glass, as described by Sir DAVID BREWSTER, in the *Philosophical Transactions* for 1815 and 1830.

When a compressed plate of this substance is examined with polarized light, it is observed to have no action on light at a perpendicular incidence; but when inclined, it shews the segments of coloured rings. This property does not belong to the plate as a whole, but is possessed by every part of it. It is therefore similar to a plate cut from a uniaxal crystal perpendicular to the axis.

I find that its action on light is like that of a *positive* crystal, while that of a plate of isinglass similarly treated would be *negative*.

The other substance which possesses similar properties is gutta percha. This substance in its ordinary state, when cold, is not transparent even in thin films; but if a thin film be drawn out gradually, it may be extended to more than double its length. It then possesses a powerful double refraction, which it retains so strongly that it has been used for polarizing light.* As one of its refractive indices is nearly the same as that of Canada balsam, while the other is very different, the common surface of the gutta percha and Canada balsam will transmit one set of rays much more readily than the other, so that a film of extended gutta percha placed between two layers of Canada balsam acts like a plate of nitre treated in the same way. That these films are in a state of constraint may be proved by heating them slightly, when they recover their original dimensions.

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* By Dr WRIGHT, I believe.

As all these permanently compressed substances have passed their limit of perfect elasticity, they do not belong to the class of elastic solids treated of in this paper; and as I cannot explain the method by which an uncrystallised body maintains itself in a state of constraint, I go on to the next case of twisting, which has more practical importance than any other. This is the case of a cylinder fixed at one end, and twisted at the other by a couple whose moment is M.

CASE II.

In this case let $\delta\theta$ be the angle of torsion at any point, then the resistance to torsion in any circular section of the cylinder is equal to the twisting force M.

The resistance at any point in the circular section is given by the second Equation of (14.)

$$q_2 = \frac{m}{2}\, r\, \frac{d\,\delta\theta}{dx}.$$

This force acts at the distance r from the axis; therefore its resistance to torsion will be $q_2 r$, and the resistance in a circular annulus will be

$$q_2 r\, 2\pi r\, dr = m\pi r^3 \frac{d\,\delta\theta}{dx}\, dr$$

and the whole resistance for the hollow cylinder will be expressed by

$$\mathrm{M} = \frac{m\pi}{4}\,\frac{d\,\delta\theta}{dx}\,(a_1^4 - a_2^4) \quad . \quad . \quad (16.)$$

$$m = 4\,\mathrm{M}\,\frac{1}{\pi\,\dfrac{\delta\theta}{b}\,(a_1^4 - a_2^4)}$$

$$m = \frac{720}{\pi^2}\,\frac{\mathrm{M}}{n}\left(\frac{b}{a_1^4 - a_2^4}\right) \quad . \quad . \quad (17.)$$

In this equation, m is the coefficient of linear elasticity; a_1 and a_2 are the radii of the exterior and interior surfaces of the hollow cylinder in inches; M is the moment of torsion produced by a weight acting on a lever, and is expressed by the product of the number of pounds in the weight into the number of inches in the lever; b is the distance of two points on the cylinder whose angular motion is measured by means of indices, or more accurately by small mirrors attached to the cylinder; n is the difference of the angle of rotation of the two indices in degrees.

This is the most accurate method for the determination of m independently of μ, and it seems to answer best with thick cylinders which cannot be used with the balance of torsion, as the oscillations are too short, and produce a vibration of the whole apparatus.

CASE III.

A hollow cylinder exposed to normal pressures only. When the pressures parallel to the axis, radius, and tangent are substituted for p_1, p_2, and p_3, Equations (10) become

$$\frac{d\,\delta\,x}{d\,x}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)(o+p+q)+\frac{1}{m}\,o \quad . \; . \quad (18.)$$

$$\frac{d\,\delta\,r}{d\,r}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)(o+p+q)+\frac{1}{m}\,p \quad . \; . \quad (19.)$$

$$\frac{d\,\delta\,(r\,\theta)}{d\,(r\,\theta)}=\frac{\delta\,r}{r}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)(o+p+q)+\frac{1}{m}\,q \quad . \; . \quad (20.)$$

By multiplying Equation (20) by r, differentiating with respect to r, and comparing this value of $\frac{d\,\delta\,r}{d\,r}$ with that of Equation (19.)

$$\frac{p-q}{r\,m}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)\left(\frac{d\,o}{d\,r}+\frac{d\,p}{d\,r}+\frac{d\,q}{d\,r}\right)-\frac{1}{m}\,\frac{d\,q}{d\,r}$$

The equation of the equilibrium of an element of the solid is obtained by considering the forces which act on it in the direction of the radius. By equating the forces which press it outwards with those pressing it inwards, we find the equation of the equilibrium of the element,

$$\frac{q-p}{r}=\frac{d\,p}{d\,r} \quad . \; . \; . \; . \quad (21.)$$

By comparing this equation with the last, we find

$$\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)\frac{d\,o}{d\,r}+\left(\frac{1}{9\,\mu}+\frac{2}{3\,m}\right)\left(\frac{d\,p}{d\,r}+\frac{d\,q}{d\,r}\right)=0$$

Integrating,

$$\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)o+\left(\frac{1}{9\,\mu}+\frac{2}{3\,m}\right)(p+q)=C_1$$

Since o, the longitudinal pressure, is supposed constant, we may assume

$$c_2=\frac{c_1-\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)0}{\frac{1}{9\,\mu}+\frac{2}{3\,m}}=(p+q)$$

$$\therefore \quad q - p = c_2 - 2p, \quad \text{therefore by (21.)}$$

$$\frac{dp}{dr} + \frac{2p}{r} = \frac{c_2}{r}$$

a linear equation, which gives

$$p = c_3 \frac{1}{r_2} + \frac{c_2}{2}$$

The coefficients c_2 and c_3 must be found from the conditions of the surface of the solid. If the pressure on the exterior cylindric surface whose radius is a_1 be denoted by h_1, and that on the interior surface whose radius is a_2 by h_2,

$$\text{then } p = h_1 \text{ when } r = a_1$$
$$\text{and } p = h_2 \text{ when } r = a_2$$

and the general value of p is

$$p = \frac{a_1^2 h_1 - a_2^2 h_2}{a_1^2 - a_2^2} - \frac{a_1^2 a_2^2}{r^2} \; \frac{h_1 - h_2}{a_1^2 - a_2^2} \quad . \; . \; . \quad (22.)$$

$$r\frac{dp}{dr} = q - p = 2\frac{a_1^2 a_2^2}{r^2} \; \frac{h_1 - h_2}{a_1^2 - a_1^2} \quad \text{by (21.)}$$

$$q = \frac{a_1^2 h_1 - a_2^2 h}{a_1^2 - a_2^2} + \frac{a_1^2 a_2^2}{r^2} \; \frac{h_1 - h_2}{a_1^2 - a_2^2} \quad . \; . \; . \quad (23.)$$

$$\mathrm{I} = b\,\omega\,(p - q) = b\,\omega \frac{a_1^2 a_2^2}{r^2} \; \frac{h_1 - h_2}{a_1^2 - a_2^2} \quad . \; . \; . \quad (24.)$$

This last equation gives the optical effect of the pressure at any point. The law of the magnitude of this quantity is the inverse square of the radius, as in Case I.; but the direction of the principal axes is different, as in this case they are parallel and perpendicular to the radius. The dark bands seen by polarized light will therefore be parallel and perpendicular to the plane of polarization, instead of being inclined at an angle of 45°, as in Case I.

By substituting in Equations (18.) and (20.), the values of p and q given in (22.) and (23.), we find that when $r = a_1$,

$$\left.\begin{aligned} \frac{\delta x}{x} &= \left(\frac{1}{9\mu}\right)\left(o + 2\frac{a_1^2 h_1 - a_2^2 h^2}{a_1^2 - a_2^2}\right) + \frac{2}{3m}\left(o - \frac{a_1^2 h_1 - a_2^2 h_2}{a_1^2 - a_2^2}\right) \\ &= o\left(\frac{1}{9\mu} + \frac{2}{3m}\right) + 2\,(h_1\,a_1^2 - h_1 a_2^2)\frac{1}{a_1^2 - a_2^2}\left(\frac{1}{9\mu} - \frac{1}{3m}\right) \end{aligned}\right\} \quad (25.)$$

$$\left.\begin{aligned} \text{When} \quad r = a_1, \frac{\delta r}{r} &= \frac{1}{9\mu}\left(o + 2\frac{a_1^2 h_1 - a_2^2 h_2}{a_1^2 - a_2^2}\right) + \frac{1}{3m}\left(\frac{a_1^2 h_1 + 3\,a_2^2\,h_1 - 4\,a_2^2\,h_2}{a_1^2 - a_2^2} - o\right) \\ &= o\left(\frac{1}{9\mu} - \frac{1}{3m}\right) + h_1\frac{1}{a_1^2 - a_2^2}\left(\frac{2\,a_1}{9\mu} + \frac{a_1^2 + 3\,a_2^1}{3m}\right) + h_2\frac{a_2}{2\,a_1 - a_2}\left(\frac{2}{9\mu} - \frac{4}{3m}\right) \end{aligned}\right\} \quad (26.)$$

From these equations it appears that the longitudinal compression of cylindric tubes is proportional to the longitudinal pressure referred to unit of surface when the lateral pressures are constant, so that for a given pressure the compression is inversely as the sectional area of the tube.

These equations may be simplified in the following cases:—

1. When the external and internal pressures are equal, or $h_1=h_2$.

2. When the external pressure is to the internal pressure as the square of the interior diameter is to that of the exterior diameter, or when $a_1^2\,h_1=a_2^2\,h_2$.

3. When the cylinder is solid, or when $a_2=0$.

4. When the solid becomes an indefinitely extended plate with a cylindric hole in it, or when a_2 becomes infinite.

5. When pressure is applied only at the plane surfaces of the solid cylinder, and the cylindric surface is prevented from expanding by being inclosed in a strong case, or when $\frac{\delta r}{r}=0$.

6. When pressure is applied to the cylindric surface, and the ends are retained at an invariable distance, or when $\frac{\delta x}{x}=0$.

1. When $h_1=h_2$, the equations of compression become

$$\left.\begin{aligned}\frac{\delta x}{x}&=\frac{1}{9\,\mu}(o+2\,h_1)+\frac{2}{3\,m}(o-h_1)\\ o&=\left(\frac{1}{9\,\mu}+\frac{2}{3\,m}\right)+2\,h_1\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)\\ \frac{\delta r}{r}&=\frac{1}{9\,\mu}(o+2\,h_1)+\frac{1}{3\,m}(h_1-o)\\ o&=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)+h_1\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)\end{aligned}\right\}\quad . \; . \quad (27.)$$

When $h_1=h_2=o$, then

$$\frac{\delta x}{x}=\frac{\delta r}{r}=\frac{h_1}{3\,\mu}.$$

The compression of a cylindrical vessel exposed on all sides to the same hydrostatic pressure is therefore independent of *m*, and it may be shewn that the same is true for a vessel of any shape.

2. When $a_1^2\,h_1=a_2^2\,h_2$,

$$\left.\begin{aligned}\frac{\delta x}{x}&=o\left(\frac{1}{9\,\mu}+\frac{2}{3\,m}\right)\\ \frac{\delta r}{r}&=\frac{1}{9\,\mu}(o)+\frac{1}{3\,m}(3\,h_1-o)\\ &=o\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)+\frac{h_1}{m}\end{aligned}\right\}\quad . \; . \quad (28.)$$

In this case, when $o=0$, the compressions are independent of μ.

3. In a solid cylinder, $a_2=0$,

$$p=q=h_1.$$

The expressions for $\frac{\delta x}{x}$ and $\frac{\delta r}{r}$ are the same as those in the first case, when $h_1=h_2$.

When the longitudinal pressure o vanishes,

$$\frac{\delta x}{x} = 2 h_1 \left(\frac{1}{9 \mu} - \frac{1}{3 m}\right)$$

$$\frac{\delta r}{r} = h_1 \left(\frac{2}{9 \mu} + \frac{1}{3 m}\right)$$

When the cylinder is pressed on the plane sides only,

$$\frac{\delta x}{x} = o \left(\frac{1}{9 \mu} + \frac{2}{3 m}\right)$$

$$\frac{\delta r}{r} = o \left(\frac{1}{9 \mu} - \frac{1}{3 m}\right)$$

4. When the solid is infinite, or when a_2 is infinite,

$$\left.\begin{aligned}
p &= h_2 + \frac{1}{r^2} a_1^2 (h_1 - h_2) \\
q &= h_2 - \frac{1}{r^2} a_1^2 (h_1 - h_2) \\
\mathrm{I} &= \omega (q - p) = -\frac{2 \omega}{r^2} a_1^2 (h_1 - h_2) \\
\frac{d x}{x} &= \frac{1}{9 \mu} (o + 2 h_2) + \frac{2}{3 m} (o - h_2) \\
&= o \left(\frac{1}{9 \mu} + \frac{2}{3 m}\right) + 2 h_2 \left(\frac{1}{9 \mu} - \frac{1}{3 m}\right) \\
\frac{d r}{r} &= \frac{1}{9 \mu} (o + 2 h_2) + \frac{1}{3 m} (4 h_2 - 3 h_2 - o) \\
&= o \left(\frac{1}{9 \mu} - \frac{1}{3 m}\right) + 2 h_2 \left(\frac{1}{9 \mu} + \frac{2}{3 m}\right) - \frac{h_1}{m}
\end{aligned}\right\} \quad . \quad . \quad (29.)$$

5. When $\delta r = 0$ in a solid cylinder,

$$\left.\begin{aligned}
&\frac{\delta x}{x} = \frac{3 o}{2 m + 3 \mu} \\
\text{6. When} \quad &\frac{\delta x}{x} = o \quad \frac{\delta r}{r} = \frac{3 h}{m + 6 \mu}
\end{aligned}\right\} \quad . \quad . \quad (30.)$$

Since the expression for the effect of a longitudinal strain is

$$\frac{\delta x}{x} = o \left(\frac{1}{9 \mu} + \frac{2}{3 m}\right)$$

if we make $\mathrm{E} = \frac{9 m \mu}{m + 6 \mu}$, then $\frac{\delta x}{x} = o \frac{1}{\mathrm{E}}$ . . . (31.)

The quantity E may be deduced from experiment on the extension of wires or rods of the substance, and μ is given in terms of m and E by the equation,

$$\mu = \frac{\mathrm{E} m}{9 m - 6 \mathrm{E}} \quad . \quad . \quad . \quad (32.)$$

and

$$\mathrm{E} = \frac{\mathrm{P} b}{s \, \delta x} \quad . \quad . \quad . \quad . \quad (33.)$$

P being the extending force, b the length of the rod, s the sectional area, and δx the elongation, which may be determined by the deflection of a wire, as in the apparatus of S' GRAVESANDE, or by direct measurement.

CASE IV.

The only known direct method of finding the compressibility of liquids is that employed by CANTON, ŒRSTED, PERKINS, AIMÉ, &c.

The liquid is confined in a vessel with a narrow neck, then pressure is applied, and the descent of the liquid in the tube is observed, so that the difference between the change of volume of the liquid and the change of internal capacity of the vessel may be determined.

Now, since the substance of which the vessel is formed is compressible, a change of the internal capacity is possible. If the pressure be applied only to the contained liquid, it is evident that the vessel will be distended, and the compressibility of the liquid will appear too great. The pressure, therefore, is commonly applied externally and internally at the same time, by means of a hydrostatic pressure produced by water compressed either in a strong vessel or in the depths of the sea.

As it does not necessarily follow, from the equality of the external and internal pressures, that the capacity does not change, the equilibrium of the vessel must be determined theoretically. ŒRSTED, therefore, obtained from POISSON his solution of the problem, and applied it to the case of a vessel of lead. To find the cubical elasticity of lead, he applied the theory of POISSON to the numerical results of TREDGOLD. As the compressibility of lead thus found was greater than that of water, ŒRSTED expected that the apparent compressibility of water in a lead vessel would be *negative*. On making the experiment the apparent compressibility was *greater* in lead than in glass. The quantity found by TREDGOLD from the extension of rods was that denoted by E, and the value of μ deduced from E alone by the formulæ of POISSON cannot be true, unless $\frac{\mu}{m} = \frac{5}{6}$; and as $\frac{\mu}{m}$ for lead is probably more than 3, the calculated compressibility is much too great.

A similar experiment was made by Professor FORBES, who used a vessel of caoutchouc. As in this case the apparent compressibility vanishes, it appears that the cubical compressibility of caoutchouc is equal to that of water.

Some who reject the mathematical theories as unsatisfactory, have conjectured that if the sides of the vessel be sufficiently thin, the pressure on both sides being equal, the compressibility of the vessel will not affect the result. The following calculations shew that the apparent compressibility of the liquid depends on the compressibility of the vessel, and is independent of the thickness when the pressures are equal.

A hollow sphere, whose external and internal radii are a_1 and a_2, is acted on by external and internal normal pressures h_1 and h_2, it is required to determine the equilibrium of the elastic solid.

The pressures at any point in the solid are:—

1. A pressure p in the direction of the radius;
2. A pressure q in the perpendicular plane.

These pressures depend on the distance from the centre, which is denoted by r.

The compressions at any point are $\frac{d\,\delta r}{d r}$ in the radial direction, and $\frac{\delta r}{r}$ in the tangent plane, the values of these compressions are:—

$$\frac{d\,\delta r}{d r} = \left(\frac{1}{9\,\mu} - \frac{1}{3\,m}\right)(p+2\,q) + \frac{1}{m}p \quad . \quad . \quad . \quad (34.)$$

$$\frac{\delta r}{r} = \left(\frac{1}{9\,\mu} - \frac{1}{3\,m}\right)(p+2\,q) + \frac{1}{m}q \quad . \quad . \quad . \quad (35.)$$

Multiplying the last equation by r, differentiating with respect to r, and equating the result with that of the first equation, we find

$$r\left(\frac{1}{9\,\mu} - \frac{1}{3\,m}\right)\left(\frac{d\,p}{d\,r} + 2\frac{d\,q}{d\,r}\right) + \frac{1}{m}\left(r\frac{d\,q}{d\,r} + q - p\right) = 0$$

Since the forces which act on the particle in the direction of the radius must balance one another, or $2\,q\,d\,r\,d\,\theta + p\,(r\,d\,\theta)^2 = (p + \frac{d\,p}{d\,r}\,d\,r)(r + d\,r)^2\,\theta$

$$\therefore \qquad q - p = \frac{r}{2}\,\frac{d\,p}{d\,r} \quad . \quad . \quad . \quad (36.)$$

Substituting this value of $q-p$ in the preceding equation, and reducing,

$$\therefore \qquad \frac{d\,p}{d\,r} + 2\frac{d\,q}{d\,r} = 0$$

Integrating, $\qquad p + 2\,q = c_1$

But $\qquad q = \frac{r}{2}\frac{d\,p}{d\,r} + p$, and the equation becomes

$$\frac{d\,p}{d\,r} + 3\frac{p}{r} + \frac{c_1}{r} = 0$$

$$\therefore \qquad p = c_2\frac{1}{r^3} + \frac{c_1}{3}$$

Since $p=h_1$ when $r=a_1$, and $p=h_2$ when $r=a_2$, the value of p at any time is found to be

$$p = \frac{a_1^{\,3}\,h_1 - a_2^{\,3}\,h_2}{a_1^{\,3} - a_2^{\,3}} - \frac{a_1^{\,3}\,a_2^{\,3}}{r^3}\,\frac{h_1 - h_2}{a^3 - b^3} \quad . \quad . \quad . \quad . \quad (37.)$$

$$q = \frac{a_1^{\,3}\,h_1 - a_2^{\,3}h_2}{a_1^{\,3} - a_2^{\,3}} - \frac{a_1^{\,3}\,a_2^{\,3}}{r^3}\,\frac{h_1 - h_2}{a_1^{\,3} - a_2^{\,3}} \quad . \quad . \quad (38.)$$

$$\frac{\delta\,\mathrm{V}}{\mathrm{V}} = 3\frac{\delta\,r}{r} = \frac{a_1^{\,3}\,h_1 - a_2^{\,3}\,h_2}{a_1^{\,3} - a_2^{\,3}}\,\frac{1}{\mu} + \frac{3}{2}\,\frac{a_1^{\,3}\,a_2^{\,3}}{r^3}\,\frac{h_1 - h_2}{a_1^{\,3} - a_2^{\,3}}\,\frac{1}{m}$$

$$\left.\begin{aligned}\text{When } r=a_1\ \frac{\delta \mathrm{V}}{\mathrm{V}} &= \frac{a_2^3 h_1 - a_2^3 h_2}{a_1^3 - a_2^3}\frac{1}{\mu} + \frac{3}{2} a_2^3 \frac{h_1 - h_2}{a_1^3 - a_2^3}\frac{1}{m} \\ &= \frac{h_1}{a_1^3 - a_2^3}\left(\frac{a_1^3}{\mu} + \frac{3 a_2^3}{2 m}\right) - \frac{h_2 a_2^3}{a_1^3 - a_2^3}\left(\frac{1}{\mu} + \frac{3}{2 m}\right)\end{aligned}\right\} \quad . \quad . \quad (39.)$$

When the external and internal pressures are equal

$$h_1 = h_2 = p = q, \text{ and } \frac{\delta \mathrm{V}}{\mathrm{V}} = \frac{h_1}{u} = \frac{h_1}{u} \quad . \quad . \quad (40.)$$

the change of internal capacity depends entirely on the cubical elasticity of the vessel, and not on its thickness or its linear elasticity.

When the external and internal pressures are inversely as the cubes of the radii of the surfaces on which they act,

$$\left.\begin{aligned} a_1^3 h_1 = a_2^3 h_2,\ p = \frac{a^3}{r^3} h_1,\ q = -\frac{1}{2}\frac{a_1^3}{r^3} h_1 \\ \frac{\delta \mathrm{V}}{\mathrm{V}} = -\frac{3}{2}\frac{a_1^3}{r^3}\frac{h_1}{m} \\ \text{when} \qquad r = a_1\ \frac{\delta \mathrm{V}}{\mathrm{V}} = -\frac{3}{2}\frac{h_1}{m}\end{aligned}\right\} \quad . \quad . \quad (41.)$$

In this case the change of capacity depends on the linear elasticity alone.

M. Regnault, in his researches on the theory of the steam engine, has given an account of the experiments which he made in order to determine with accuracy the compressibility of mercury.

He considers the mathematical formulæ very uncertain, because the theories of molecular forces from which they are deduced are probably far from the truth; and even were the equations free from error, there would be much uncertainty in the ordinary method by measuring the elongation of a rod of the substance, for it is difficult to ensure that the material of the rod is the same as that of the hollow sphere.

He has, therefore, availed himself of the results of M. Lamé for a hollow sphere in three different cases, in the first of which the pressure acts on the interior and exterior surface at the same time, while in the other two cases the pressure is applied to the exterior or interior surface alone. Equation (39.) becomes in these cases,—

1. When $h_1 = h_2\ \frac{\delta \mathrm{V}}{\mathrm{V}} = \frac{h_1}{\mu}$ and the compressibility of the enclosed liquid being μ_2, and the apparent diminution of volume $\delta' \mathrm{V}$, $\frac{\delta' \mathrm{V}}{\mathrm{V}} = h_1\left(\frac{1}{\mu_2} - \frac{1}{\mu}\right)$ . . (42.)

2. When $h_1 = o,\ \frac{\delta \mathrm{V}}{\mathrm{V}} = \frac{\delta' \mathrm{V}}{\mathrm{V}} = -h_2 \frac{a_2^3}{a_1^3 - a_2^3}\left(\frac{1}{\mu} + \frac{3}{2 m}\right)$ . . . . (43.)

3. When $h_2 = o,\ \frac{\delta \mathrm{V}}{\mathrm{V}} = \frac{h_1}{a_1^3 - a_2^3}\left(\frac{a_1^3}{\mu} + \frac{3}{2}\frac{a_2^3}{m}\right)$

$$\frac{\delta' \mathrm{V}}{\mathrm{V}} = \frac{h_1}{a_1^3 - a_2^3}\left(\frac{a_1^3}{\mu} + \frac{3}{2}\frac{a_2^3}{m} + (a_2^3 - a_1^3)\frac{1}{\mu_2}\right)$$

M. LAMÉ's equations differ from these only in assuming that $\mu = \frac{5}{6} m$. If this assumption be correct, then the coefficients μ, m, and μ_2, may be found from two of these equations; but since one of these equations may be derived from the other two, the *three* coefficients cannot be found when μ is supposed independent of m. In Equations (39.), the quantities which may be varied at pleasure are h_1 and h_2, and the quantities which may be deduced from the apparent compressions are,

$$c_1 = \left(\frac{1}{\mu} + \frac{3}{2\,m}\right) \text{ and } \left(\frac{1}{\mu} - \frac{1}{\mu_2}\right) = c_2$$

therefore some independent equation between these quantities must be found, and this cannot be done by means of the sphere alone; some other experiment must be made on the liquid, or on another portion of the substance of which the vessel is made.

The value of μ_2, the elasticity of the liquid, may be previously known.

The linear elasticity m of the vessel may be found by twisting a rod of the material of which it is made;

Or, the value of E may be found by the elongation or bending of the rod, and $\frac{1}{\mathrm{E}} = \frac{1}{9\,\mu} + \frac{2}{3\,m}$.

We have here five quantities, which may be determined by experiment.

$$\left.\begin{array}{lll} (43.)\ \ 1. & c_1 = \left(\frac{1}{\mu} + \frac{3}{2\,m}\right) & \text{by external pressure} \\ (42.)\ \ 2. & c_2 = \left(\frac{1}{\mu} - \frac{1}{\mu_2}\right) & \text{equal pressures} \end{array}\right\} \text{on sphere.}$$

$$\begin{array}{lll} (31.)\ \ 3. & \frac{1}{\mathrm{E}} = \left(\frac{1}{9\mu} + \frac{2}{3\,m}\right) & \text{by elongation of a rod.} \\ (17.)\ \ 4. & m & \text{by twisting the rod.} \\ \ \ 5. & \mu_2 & \text{the elasticity of the liquid.} \end{array}$$

When the elastic sphere is solid, the internal radius a_1 vanishes, and $h_2 = p = q$, and $\frac{\delta\mathrm{V}}{\mathrm{V}} = \frac{h_2}{\mu}$.

When the case becomes that of a spherical cavity in an infinite solid, the external radius a_2 becomes infinite, and

$$\left.\begin{array}{l} p = h_2 + \frac{a_1^3}{r^3}(h_1 - h_2) \\ q = h_2 - \frac{1}{2}\frac{a_1^3}{r^3}(h_1 - h_2) \\ \frac{\delta r}{r} = h_2 \frac{1}{3\,\mu} - \frac{1}{2}\frac{a_1^3}{r^3}(h_1 - h_2)\frac{1}{m} \\ \frac{\delta\mathrm{V}}{\mathrm{V}} = \frac{h_2}{\mu} - \frac{1}{2}\frac{h_1 - h_2}{m} \end{array}\right\} \quad . \quad . \quad . \quad (46.)$$

The effect of pressure on the surface of a spherical cavity on any other part of an elastic solid is therefore inversely proportional to the cube of its distance from the centre of the cavity.

When one of the surfaces of an elastic hollow sphere has its radius rendered invariable by the support of an incompressible sphere, whose radius is a_1, then

$$\frac{\delta r}{r}=0, \quad \text{when } r=a_1$$

$$\left.\begin{aligned}
\therefore \quad p&=h_2\frac{3a_2^3\mu}{2a_1^3m+3a_2^3\mu}+h_2\frac{a_1^3a_2^3}{r^3}\frac{2m}{2a_1^3m+3a_2^3\mu}\\
q&=h_2\frac{3a_2^3\mu}{2a_1^3m+3a_2^3\mu}-h_2\frac{a_1^3a_2^3}{r^3}\frac{m}{2a_1^3m+3a_2^3\mu}\\
\frac{\delta r}{r}&=h_2\frac{a_2^3}{2a_1^3m+3a_2^3\mu}-h_2\frac{a_1^3a_2^3}{r^3}\frac{1}{2a_1^3m+3a_2^3\mu}\\
\text{When } r=a_2 \quad \frac{\delta V}{V}&=h_2\frac{3a_2^3-3a_1^3}{2a_1^3m+3a_2^3\mu}
\end{aligned}\right\} \quad . \quad . \quad . \quad (45.)$$

CASE V.

On the equilibrium of an elastic beam of rectangular section uniformly bent.

By supposing the bent beam to be produced till it returns into itself, we may treat it as a hollow cylinder.

Let a rectangular elastic beam, whose length is $2\pi c$, be bent into a circular form, so as to be a section of a hollow cylinder, those parts of the beam which lie towards the centre of the circle will be longitudinally compressed, while the opposite parts will be extended.

The expression for the tangential compression is therefore

$$\frac{\delta r}{r}=\frac{c-r}{c}.$$

Comparing this value of $\frac{\delta r}{r}$ with that of Equation (20.)

$$\frac{c-r}{r}=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)(o+p+q)+\frac{q}{m}$$

and by (21.) $$q=p+r\frac{dp}{dr}.$$

By substituting for q its value, and dividing by $r\left(\frac{1}{9\mu}+\frac{1}{3m}\right)$, the equation becomes

$$\frac{dp}{dr}+\frac{2m+3\mu}{m+6\mu}\frac{p}{r}=\frac{9m\mu-(m-3\mu)o}{(m+6\mu)r}-\frac{9m\mu}{(m+6\mu)c}$$

a linear differential equation, which gives

$$p=C_1r^{-\frac{2m+3\mu}{m+6\mu}}-\frac{3m\mu}{m+3\mu}\frac{r}{c}+\frac{9\mu m-(m-3\mu)o}{2m+3\mu} \quad . \quad . \quad (46.)$$

C_1 may be found by assuming that when $r = a_1$ $p = h_1$, and q may be found from p by Equation (21.)

As the expressions thus found are long and cumbrous, it is better to use the following approximations:—

$$q = -\left(\frac{9m\mu}{m+6\mu}\right)\frac{y}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (47.)$$

$$p = \left(\frac{9m\mu}{m+6\mu}\right)\frac{1}{2c}\left(\frac{c^2-a^2}{y-c}+c+y\right) \quad . \quad (48.)$$

In these expressions a is half the depth of the beam, and y is the distance of any part of the beam from the neutral surface, which in this case is a cylindric surface, whose radius is c.

These expressions suppose c to be large compared with a, since most substances break when $\frac{a}{c}$ exceeds a certain small quantity.

Let b be the breadth of the beam, then the force with which the beam resists flexure $=$M.

$$M = \int b\,y\,q = \frac{q\,m\mu}{m+6\mu}\frac{b}{c}\,\frac{a^3}{3} = E\,\frac{a^3 b}{3c} \quad . \quad . \quad . \quad (49.)$$

which is the ordinary expression for the stiffness of a rectangular beam.

The stiffness of a beam of any section, the form of which is expressed by an equation between x and y, the axis of x being perpendicular to the plane of flexure, or the osculating plane of the axis of the beam at any point, is expressed by

$$M\,c = E\int y^2\,dx, \quad . \quad . \quad . \quad (50.)$$

M being the moment of the force which bends the beam, and c the radius of the circle into which it is bent.

CASE VI.

At the meeting of the British Association in 1839, Mr JAMES NASMYTH described his method of making concave specula of silvered glass by bending.

A circular piece of silvered plate-glass was cemented to the opening of an iron vessel, from which the air was afterwards exhausted. The mirror then became concave, and the focal distance depended on the pressure of the air.

BUFFON proposed to make burning-mirrors in this way, and to produce the partial vacuum by the combustion of the air in the vessel, which was to be effected by igniting sulphur in the interior of the vessel by means of a burning-glass. Although sulphur evidently would not answer for this purpose, phosphorus might; but the simplest way of removing the air is by means of the air-pump. The mirrors which were actually made by BUFFON, were bent by means of a screw acting on the centre of the glass.

To find an expression for the curvature produced in a flat, circular, elastic plate, by the difference of the hydrostatic pressures which act on each side of it,—

Let t be the thickness of the plate, which must be small compared with its diameter.

Let the form of the middle surface of the plate, after the curvature is produced, be expressed by an equation between r, the distance of any point from the axis, or normal to the centre of the plate, and x the distance of the point from the plane in which the middle of the plate originally was, and let $ds=\sqrt{(dx)^2+(dr^2)}$.

Let h_1 be the pressure on one side of the plate, and h_2 that on the other.

Let p and q be the pressures in the plane of the plate at any point, p acting in the direction of a tangent to the section of the plate by a plane passing through the axis, and q acting in the direction perpendicular to that plane.

By equating the forces which act on any particle in a direction parallel to the axis, we find

$$tp\frac{dr}{ds}\,\frac{dx}{ds}+tr\frac{dp}{ds}\,\frac{dx}{ds}+trp\frac{d^2x}{ds^2}+r(h_1-h_2)\frac{dr}{ds}=0$$

By making $p=0$ when $r=0$ in this equation,

$$p=-\frac{r}{2t}\,\frac{ds}{dx}(h_1-h_2) \quad . \quad . \quad . \quad (51.)$$

The forces perpendicular to the axis are

$$tp\left(\frac{dr}{ds}\right)^2+tr\frac{dp}{ds}\,\frac{dr}{ds}+trp\frac{d^2r}{ds^2}-(h_1-h_2)r\frac{dx}{ds}-qt=0$$

Substituting for p its value, the equation gives

$$q=-\frac{(h_1-h_2)}{t}r\left(\frac{dr}{ds}\,\frac{dr}{dx}+\frac{dx}{ds}\right)+\frac{(h_1-h_2)}{2t}r^2\left(\frac{dr}{dx}\,\frac{ds}{dx}\,\frac{d^2x}{ds^2}-\frac{ds}{dx}\,\frac{d^2r}{ds^2}\right) \quad . \quad . \quad . \quad (52.)$$

The equations of elasticity become

$$\frac{d\,\delta s}{ds}=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(p+q+\frac{h_1+h_2}{2}\right)+\frac{p}{m}$$

$$\frac{\delta r}{r}=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(p+q+\frac{h_1+h_2}{2}\right)+\frac{q}{m}$$

Differentiating $\dfrac{d\,\delta r}{dr}=\dfrac{d}{dr}\left(\dfrac{\delta r}{r}\,r\right)$, and in this case

$$\frac{d\,\delta r}{dr}=1-\frac{dr}{ds}+\frac{dr}{ds}\,\frac{d\,\delta s}{ds}$$

By a comparison of these values of $\dfrac{d\,\delta r}{ds}$

$$\left(1-\frac{dr}{ds}\right)\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(p+q+\frac{h_1+h_2}{2}\right)+\frac{q}{m}+\frac{dr}{ds}\,\frac{p}{m}+r\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(\frac{dp}{dr}+\frac{dq}{dr}\right)$$

$$+\frac{r}{m}\,\frac{dq}{dr}+\frac{dr}{ds}-1=0.$$

To obtain an expression for the curvature of the plate at the vertex, let a be the radius of curvature, then, as an approximation to the equation of the plate, let

$$x = \frac{r^2}{2a}.$$

By substituting the value of x in the values of p and q, and in the equation of elasticity, the approximate value of a is found to be

$$a = \frac{t}{h_1 - h_2} \frac{(h_1 + h_2)\left(\frac{1}{9\mu} - \frac{1}{3m}\right) - 2}{10\left(\frac{1}{9\mu} - \frac{1}{3m}\right) - \frac{9}{m}}$$

$$a = \frac{t}{h_1 - h_2} \frac{-18m\mu}{10m + 51\mu} + t\frac{h_1 + h_2}{h_1 - h_2} \frac{m - 3\mu}{10m + 51\mu} \quad . \quad . \quad . \quad (53.)$$

Since the focal distance of the mirror, or $\frac{a}{2}$, depends on the difference of pressures, a telescope on Mr NASMYTH's principle would act as an aneroid barometer, the focal distance varying inversely as the pressure of the atmosphere.

CASE VII.

To find the conditions of torsion of a cylinder composed of a great number of parallel wires bound together without adhering to one another.

Let x be the length of the cylinder, a its radius, r the radius at any point, $\delta\theta$ the angle of torsion, M the force producing torsion, δx the change of length, and P the longitudinal force. Each of the wires becomes a helix whose radius is r_1, its angular rotation $\delta\theta$, and its length along the axis $x - \delta\theta$.

Its length is therefore $\sqrt{(r\delta\theta)^2 + x\left(1 - \frac{\delta x}{x}\right)^2}$

and the tension is $= \mathrm{E}\left(1 - \sqrt{\left(1 - \frac{\delta x}{x}\right)^2 + r^2\left(\frac{\delta\theta}{x}\right)^2}\right)$

This force, resolved parallel to the axis, is

$$\frac{d}{d\theta} \frac{d}{dr} \mathrm{P} = \mathrm{E}\left(\frac{1}{\sqrt{\left(1 - \frac{\delta x}{x}\right)^2 + r^2\left(\frac{\delta\theta}{x}\right)^2}} - 1\right)$$

and since $\frac{\delta x}{x}$ and $r\frac{\delta\theta}{x}$ are small, we may assume

$$\frac{d}{d\theta} \frac{d}{dr} \mathrm{P} = \mathrm{E}\left(\frac{\delta x}{x} - \frac{r^2}{2}\left(\frac{\delta\theta}{x}\right)^2\right)$$

$$\mathrm{P} = \pi\mathrm{E}\left(r^2\frac{\delta x}{x} - \frac{r^4}{4}\left(\frac{\delta\theta}{x}\right)^2\right) \quad . \quad . \quad . \quad (54.)$$

The force, when resolved in the tangential direction, is approximately

$$\frac{1}{r^2} \frac{d}{d\theta} \frac{d}{dx} \mathrm{M} = \mathrm{E}\left(r\frac{\delta\theta}{x} \frac{\delta x}{x} - \frac{r^3}{2}\left(\frac{\delta\theta}{x}\right)^3\right)$$

$$\mathrm{M} = \pi\mathrm{E}\left(\frac{r^4}{2} \frac{\delta\theta}{x} \frac{\delta x}{x} - \frac{r^6}{6}\left(\frac{\delta\theta}{x}\right)^3\right) \quad . \quad . \quad (55.)$$

By eliminating $\frac{\delta x}{x}$ between (54.) and (55.) we have

$$\mathrm{M}=\frac{r^2}{2}\,\frac{\delta\theta}{x}\,\mathrm{P}-\mathrm{E}\,\pi\frac{r^6}{24}\left(\frac{\delta\theta}{x}\right)^3 \quad . \quad . \quad . \quad (56.)$$

When P=0, M depends on the sixth power of the radius and the cube of the angle of torsion, when the cylinder is composed of separate filaments.

Since the force of torsion for a homogeneous cylinder depends on the fourth power of the radius and the first power of the angle of torsion, the torsion of a wire having a fibrous texture will depend on both these laws.

The parts of the force of torsion which depend on these two laws may be found by experiment, and thus the difference of the elasticities in the direction of the axis and in the perpendicular directions may be determined.

A calculation of the force of torsion, on this supposition, may be found in YOUNG'S *Mathematical Principles of Natural Philosophy;* and it is introduced here to account for the variations from the law of Case II., which may be observed in a twisted rod.

CASE VIII.

It is well known that grindstones and fly-wheels are often broken by the centrifugal force produced by their rapid rotation. I have therefore calculated the strains and pressure acting on an elastic cylinder revolving round its axis, and acted on by the centrifugal force alone.

The equation of the equilibrium of a particle (see Equation (21.)), becomes

$$q-p=r\frac{dp}{dr}-\frac{4\pi^2 k}{g t^2}r^2;$$

where q and p are the tangential and radial pressures, k is the weight in pounds of a cubic inch of the substance, g is twice the height in inches that a body falls in a second, t is the time of revolution of the cylinder in seconds.

By substituting the value of g and $\frac{dq}{dr}$ in Equations (19.), (20.), and neglecting o,

$$0=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(3\frac{dp}{dr}-2\frac{4\pi^2 k}{g t^2}r+r\frac{d^2 p}{dr^2}\right)+\frac{1}{m}\left(3\frac{dp}{dr}-3\frac{4\pi^2 k}{g t^2}r+r\frac{d^2 p}{dr^2}\right)$$

which gives

$$\left.\begin{aligned} p&=c_1\frac{1}{r^2}+\frac{\pi^2 k}{2 g t^2}\left(2+\frac{\mathrm{E}}{m}\right)r^2+c_2 \\ \therefore\quad q-p&=-c_1\frac{1}{r^2}+\frac{\pi^2 k}{2 g t}\left(-4+\frac{2\mathrm{E}}{m}\right)r^2 \\ q&=-c_1\frac{1}{r^2}+\frac{\pi^2 k}{2 g t^2}\left(-2+\frac{3\mathrm{E}}{m_3}\right)r^2+c_2 \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad (57.)$$

If the radii of the surfaces of the hollow cylinder be a_1 and a_2, and the pressures acting on them h_1 and h_2, then the values of c_1 and c_2 are

$$\left.\begin{aligned} c_1 &= a_1^2\, a_2^2 \frac{\pi^2 k}{2 g t^2}\left(2+\frac{\mathrm{E}}{m}\right) - a_1^2\, a_2^2 \frac{h_1-h_2}{a_1^2-a_2^2} \\ c_2 &= \frac{a_1^2 h_1 - a_2^2 h_2}{a_1^2-a_2^2} - (a_1^2+a_2^2)\frac{\pi^2 k}{2 g t^2}\left(2+\frac{\mathrm{E}}{m}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (58.)$$

When $a_2=0$, as in the case of a solid cylinder, $c_1=0$, and

$$c_2 = h_1 - a_1^2 \frac{\pi^2 k}{2 g t^2}\left(2+\frac{\mathrm{E}}{m}\right)$$

$$q = h_1 + \frac{\pi^2 k}{2 g t^2}\left\{2(r^2+a_1^2) + \frac{\mathrm{E}}{m}(3 r^2 - a_1^2)\right\} \quad . \quad . \quad . \quad . \quad (59.)$$

When $h_1=0$, and $r=a_1$,

$$q = \frac{\pi^2 k a^2}{g t^2}\left(\frac{\mathrm{E}}{m} - 2\right) \quad . \quad . \quad . \quad . \quad (60.)$$

When q exceeds the tenacity of the substance in pounds per square inch, the cylinder will give way; and by making q equal to the number of pounds which a square inch of the substance will support, the velocity may be found at which the bursting of the cylinder will take place.

Since $\mathrm{I} = b\,\omega\,(q-p) = \frac{\pi^2 k\,\omega}{g\,t}\left(\frac{\mathrm{E}}{m} - 2\right) b\,r^2$, a transparent revolving cylinder, when polarized light is transmitted parallel to the axis, will exhibit rings whose diameters are as the square roots of an arithmetical progression, and brushes parallel and perpendicular to the plane of polarization.

CASE IX.

A hollow cylinder or tube is surrounded by a medium of a constant temperature while a liquid of a different temperature is made to flow through it. The exterior and interior surfaces are thus kept each at a constant temperature till the transference of heat through the cylinder becomes uniform.

Let v be the temperature at any point, then when this quantity has reached its limit,

$$\frac{r\,d v}{d r} = c_1$$

$$x = c_1 \log r + c_2 \quad . \quad . \quad . \quad . \quad (61.)$$

Let the temperatures at the surfaces be θ_1 and θ_2, and the radii of the surfaces a_1 and a_2, then

$$c_1 = \frac{\theta_1 - \theta_2}{\log a_1 - \log a_2} \quad c_2 = \frac{\log a_1\, \theta_2 - \log a_2\, \theta_1}{\log a_1 - \log a_2}$$

Let the coefficient of linear dilatation of the substance be c_3, then the proportional dilatation at any point will be expressed by $c_3\, v$, and the equations of elasticity (18.), (19.), (20.), become

$$\frac{d\,\delta x}{d x} = \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(o+p+q) + \frac{o}{m} - c_3\, v$$

$$\frac{d\,\delta r}{d r}=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)(o+p+q)+\frac{p}{m}-c_3 v$$

$$\frac{\delta r}{r}=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)(o+p+q)+\frac{q}{m}-c_3 v$$

The equation of equilibrium is

$$q=p+r\frac{dp}{dr} \quad . \quad . \quad (21.)$$

and since the tube is supposed to be of a considerable length

$$\frac{d\,\delta x}{dx}=c_4 \text{ a constant quantity.}$$

From these equations we find that

$$o=\frac{c_4+c_3 v-\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(2p+r\frac{dp}{dr}\right)}{\frac{1}{9\mu}+\frac{2}{3m}}$$

and hence $v=c_1 \log r+c_2$, p may be found in terms of r.

$$p=\left(\frac{2}{9\mu}+\frac{1}{3m}\right)^{-1} c_1 c_3 \log r+c_5\frac{1}{r^2}+c_6$$

Hence $$q=\left(\frac{2}{9\mu}+\frac{1}{3m}\right)^{-1} c_1 c_3 \log r-c_5\frac{1}{r^2}+c_6+\left(\frac{2}{9\mu}+\frac{1}{3m}\right)c_1 c_3$$

Since $$\mathrm{I}=b\,\omega\,(q+p)=b\,\omega\left(\frac{2}{9\mu}+\frac{1}{3m}\right)^{-1} c_1 c_3-2\,b\,\omega\,c_5\frac{1}{r^2}$$

the rings seen in this case will differ from those described in Case III. only by the addition of a constant quantity.

When no pressures act on the exterior and interior surfaces of the tube $h_1=h_2=0$, and

$$62.\quad\begin{cases} p=\left(\frac{2}{9\mu}+\frac{1}{3m}\right)^{-1} c_1 c_3\left\{\log r+\frac{a_1^2 a_2^2}{r^2}\frac{\log a_1-\log a_2}{a_1^2-a_2^2}+\frac{a_1^2\log a_1-a_2^2\log a_2}{a_1^2-a_2^2}\right\} \\ q=\left(\frac{2}{9\mu}+\frac{1}{3m}\right)^{-1} c_1 c_3\left\{\log r-\frac{a_1^2 a_2^2}{r^2}\frac{\log a_1-\log a_2}{a_1-a_2}+\frac{a_1^2\log a_1-a_2^2\log a_2}{a_1^2-a_2^2}+1\right\} \\ \mathrm{I}=b\left(\frac{2}{9\mu}+\frac{1}{3m}\right)^{-1} c_1 c_3\,\omega\left\{1-2\frac{a_1^2 a_2^2}{r^2}\frac{\log a_1-\log a_2}{a_1^2-a_2^2}\right\} \end{cases}$$

There will, therefore, be no action on polarized light for the ring whose radius is r when

$$r^2=2\frac{a_1^2 a_2^2}{a_1^2-a_2^2}\log\frac{a_1}{a_2}$$

CASE X.

Sir David Brewster has observed (*Edinburgh Transactions*, vol. viii.), that when a solid cylinder of glass is suddenly heated at the cylindric surface a polarizing force is developed, which is at any point proportional to the square of the distance from the axis of the cylinder; that is to say, that the difference of retardation of the oppositely polarized rays of light is proportional to the square of the radius r, or

$$\mathrm{I}=b\,c_1\,\omega\,r^2=b\,\omega\,(q-p)=b\,\omega\,r\frac{d\,p}{d\,r}$$

$$\therefore\frac{d\,p}{d\,r}=c_1\,r\;\therefore\;p=\frac{c_1}{2}\,r^2+c_2$$

Since if a be the radius of the cylinder, $p=o$ when $r=a$,

$$p=\frac{c_1}{2}\,(r^2-a^2)$$

Hence $$q=\frac{c_1}{2}\,(3\,r^2-a^2)$$

By substituting these values of p and q in equations (19) and (20), and making $\frac{d}{d\,r}\frac{\delta\,r}{r}r=\frac{d\,\delta\,r}{d\,r}$, I find,

$$v=\frac{2\,c_1}{c_3}\left(\frac{1}{9\,\mu}+\frac{2}{3\,m}\right)r^2+c_4 \tag{63.}$$

c_4 being the temperature of the axis of the cylinder, and c_3 the coefficient of linear expansion for glass.

CASE XI.

Heat is passing uniformly through the sides of a spherical vessel, such as the ball of a thermometer, it is required to determine the mechanical state of the sphere. As the methods are nearly the same as in Case IX., it will be sufficient to give the results, using the same notation.

$$r^2\frac{d\,v}{d\,v}=c_1\;\therefore\;v=c_2-\frac{c_1}{r}$$

$$c_1=a_1\,a_2\,\frac{\theta_1-\theta_2}{a_1-a_2}\qquad c_2=\frac{\theta_1\,a_1-\theta_2\,a_2}{a_1-a_2}$$

$$p=c_4\,\frac{1}{r^3}+\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_1\,c_3=\frac{1}{r}-c_5$$

When $h_1=h_2=0$ the expression for p becomes

$$p=\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_3\,(\theta_1-\theta_2)\left\{\frac{a_1{}^3\,a_2{}^3}{a_1{}^3-a_2{}^3}\,\frac{1}{r^3}+\frac{a_1\,a_2}{a_1-a_2}\,\frac{1}{r}-a_1\,a^2\,\frac{a_1{}^2-a_2{}^2}{(a_1-a_2)\,(a_1{}^3-a_2{}^3)}\right\} \tag{64.}$$

From this value of p the other quantities may be found, as in Case IX., from the equations of Case IV.

CASE XII.

When a long beam is bent into the form of a closed circular ring (as in Case V.), all the pressures act either parallel or perpendicular to the direction of the length of the beam, so that if the beam were divided into planks, there would be no tendency of the planks to slide on one another.

But when the beam does not form a closed circle, the planks into which it may be supposed to be divided will have a tendency to slide on one another, and the amount of sliding is determined by the linear elasticity of the substance. The deflection of the beam thus arises partly from the bending of the whole beam, and partly from the sliding of the planks; and since each of these deflections is small compared with the length of the beam, the total deflection will be the sum of the deflections due to bending and sliding.

Let $$A = \mathrm{M}\,c = \mathrm{E}\int x\,y^2\,dy \quad . \quad . \quad (65.)$$

A is the stiffness of the beam as found in Case V., the equation of the transverse section being expressed in terms of x and y, y being measured from the neutral surface.

Let a horizontal beam, whose length is $2\,l$, and whose weight is $2\,w$, be supported at the extremities and loaded at the middle with a weight W.

Let the deflection at any point be expressed by $\delta_1\,y$, and let this quantity be small compared with the length of the beam.

At the middle of the beam, $\delta_1\,y$ is found by the usual methods to be

$$\delta_1\,y = \frac{1}{A}\left(\frac{5}{24}\,l^3\,w + \frac{1}{6}\,l^3\,W\right) \quad . \quad . \quad . \quad . \quad (66.)$$

Let $$B = \frac{m}{2}\int x\,dy = \frac{m}{2}\,(\text{sectional area}). \quad . \quad . \quad (67.)$$

B is the resistance of the beam to the sliding of the planks. The deflection of the beam arising from this cause is

$$\delta_2\,y = \frac{l}{2\,B}\,(w + W) \quad . \quad . \quad . \quad . \quad . \quad . \quad (68.)$$

The quantity is small compared with $\delta_1\,y$, when the depth of the beam is small compared with its length.

The whole deflection $\Delta\,y = \delta_1\,y + \delta_2\,y$

$$\Delta\,y = \frac{l^3}{6\,A}\left(\frac{5}{4}\,w + W\right) + \frac{l}{2\,B}\,(w + W)$$

$$\Delta\,y = w\left(\frac{5}{24}\,\frac{l^3}{A} + \frac{1}{2}\,\frac{l}{B}\right) + W\left(\frac{l^3}{6\,A} + \frac{1}{2}\,\frac{l}{B}\right) \quad . \quad (69.)$$

CASE XIII.

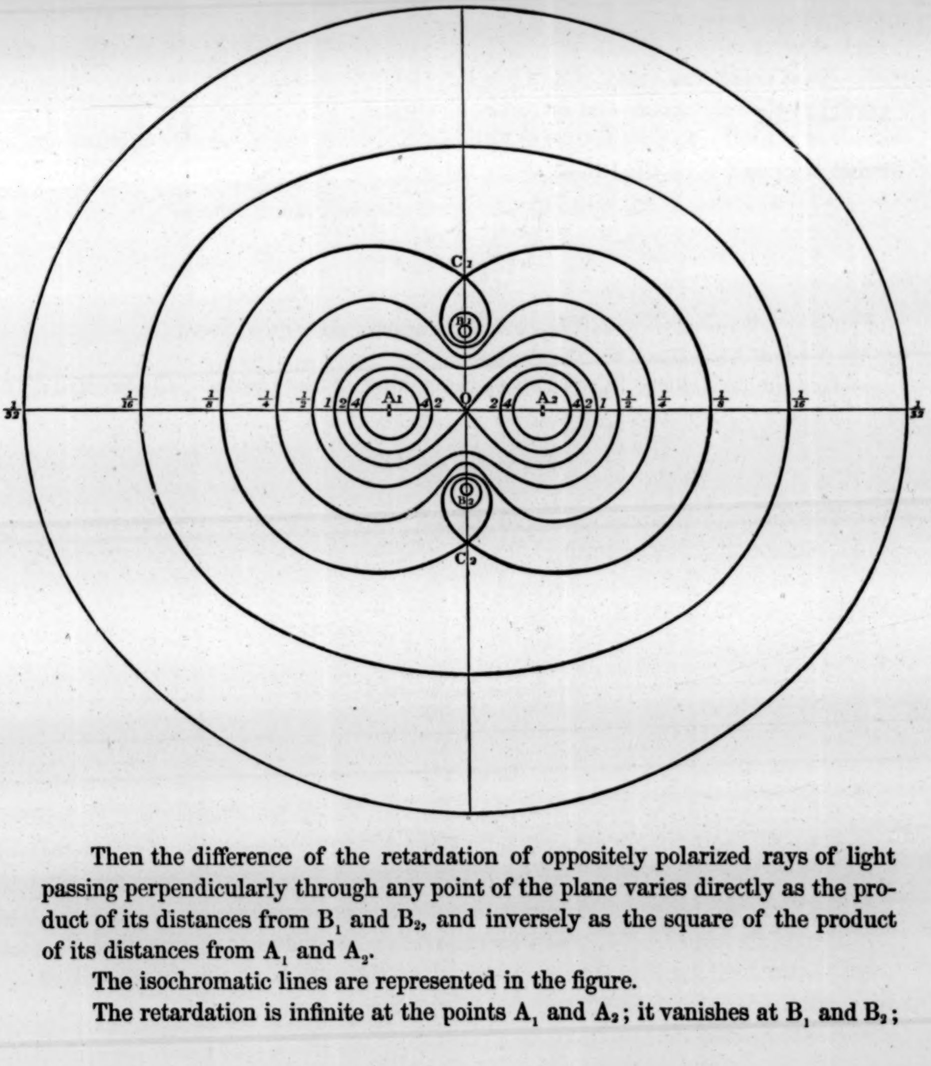
When the values of the compressions at any point have been found, when two different sets of forces act on a solid separately, the compressions, when the forces act at the same time, may be found by the composition of compressions, because the small compressions are independent of one another.

It appears from Case I., that if a cylinder be twisted as there described, the compressions will be inversely proportional to the square of the distance from the centre.

If *two* cylindric surfaces, whose axes are perpendicular to the plane of an indefinite elastic plate, be equally twisted in the same direction, the resultant compression in any direction may be found by adding the compression due to each resolved in that direction.

The result of this operation may be thus stated geometrically. Let A_1 and A_2 (fig. 1.) be the centres of the twisted cylinders. Join A_1 A_2, and bisect A_1 A_2 in O. Draw OBC at right angles, and cut off OB_1 and OB_2 each equal to OA_1.

Fig. 1.



Then the difference of the retardation of oppositely polarized rays of light passing perpendicularly through any point of the plane varies directly as the product of its distances from B_1 and B_2, and inversely as the square of the product of its distances from A_1 and A_2.

The isochromatic lines are represented in the figure.

The retardation is infinite at the points A_1 and A_2; it vanishes at B_1 and B_2;

and if the retardation at o be taken for unity, the isochromatic curves 2, 4, surround A_1 and A_2; that in which the retardation is unity has two loops, and passes through O; the curves $\frac{1}{2}$, $\frac{1}{4}$ are continuous, and have points of contrary flexure; the curve $\frac{1}{8}$ has multiple points at C_1 and C_2, where $A_1 C_1 = A_1 A_2$, and two loops surrounding B_1 and B_2; the other curves, for which $I=\frac{1}{16}, \frac{1}{32}$, &c., consists each of two ovals surrounding B_1 and B_2, and an exterior portion surrounding all the former curves.

I have produced these curves in the jelly of isinglass described in Case I. They are best seen by using circularly polarized light, as the curves are then seen without interruption, and their resemblance to the calculated curves is more apparent. To avoid crowding the curves toward the centre of the figure, I have taken the values of I for the different curves, not in an arithmetical, but in a geometrical progression, ascending by powers of 2.

CASE XIV.

On the determination of the pressures which act in the interior of transparent solids, from observations of the action of the solid on polarized light.

Sir DAVID BREWSTER has pointed out the method by which polarized light might be made to indicate the strains in elastic solids; and his experiments on bent glass confirm the theories of the bending of beams.

The phenomena of heated and unannealed glass are of a much more complex nature, and they cannot be predicted and explained without a knowledge of the laws of cooling and solidification, combined with those of elastic equilibrium.

In Case X. I have given an example of the inverse problem, in the case of a cylinder in which the action on light followed a simple law; and I now go on to describe the method of determining the pressures in a general case, applying it to the case of a triangle of unannealed plate-glass.

The lines of equal intensity of the action on light are seen without interruption, by using circularly polarized light. They are represented in fig. 2, where A, BBB, DDD are the neutral points, or points of no action on light, and CCC, EEE are the points where that action is greatest; and the intensity of the action at any other point is determined by its position with respect to the isochromatic curves.

The direction of the principal axes of pressure at any point is found by transmitting plane polarized light, and analysing it in the plane perpendicular to that of polarization. The light is then restored in every part of the triangle, except in those points at which one of the principal axes is parallel to the plane of polarization. A dark band formed of all these points is seen, which shifts its position as the triangle is turned round in its own plane. Fig. 3 represents these

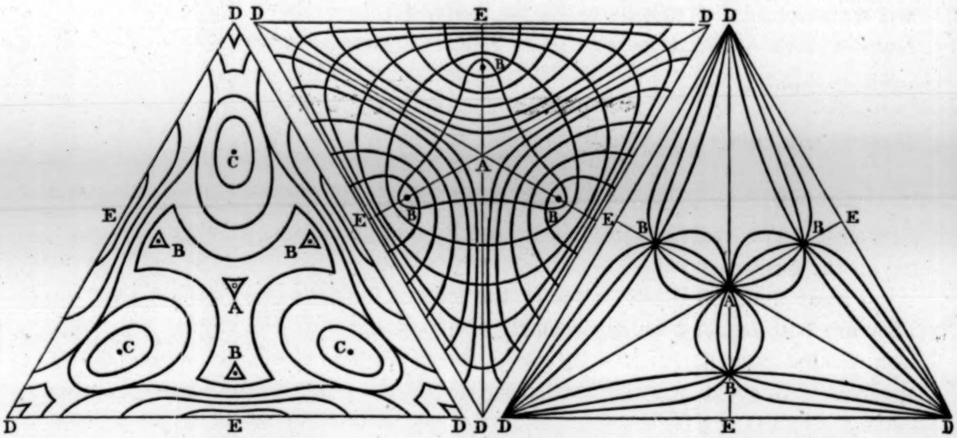


Fig. 2. Fig. 4. Fig. 3.

curves for every fifteenth degree of inclination. They correspond to the lines of equal variation of the needle in a magnetic chart.

From these curves others may be found which shall indicate, by their own direction, the direction of the principal axes at any point. These curves of direction of compression and dilatation are represented in fig. 4; the curves whose direction corresponds to that of *compression* are concave toward the centre of the triangle, and intersect at right angles the curves of dilatation.

Let the isochromatic lines in fig. 2 be determined by the equation

$$\phi_1(x_1 y) = \mathrm{I}\frac{1}{z} = \omega(q-p)\frac{1}{z}$$

where I is the difference of retardation of the oppositely polarized rays, and q and p the pressure in the principal axes at any point, z being the thickness of the plate.

Let the lines of equal inclination be determined by the equation

$$\phi_2(x_1 y) = \tan\theta$$

θ being the angle of inclination of the principal axes; then the differential equation of the curves of direction of compression and dilatation (fig. 4) is

$$\phi_2(x_1 y) = \frac{dy}{dx}$$

By considering any particle of the plate as a portion of a cylinder whose axis passes through the centre of curvature of the curve of compression, we find

$$q-p = r\frac{dp}{dr} \quad . \quad . \quad (21.)$$

Let R denote the radius of curvature of the curve of compression at any point, and let S denote the length of the curve of dilatation at the same point,

$$\phi_3(x_1 y) = \mathrm{R} \qquad \phi_4(x_1 y) = \mathrm{S}$$

$$q-p = \mathrm{R}\frac{dp}{ds}$$

and since $(q-p)$, R and S are known, and since at the surface, where $\phi_5(x_1 y)=0$, $p=0$, all the data are given for determining the absolute value of p by integration.

Though this is the best method of finding p and q by graphic construction, it is much better, when the equations of the curves have been found, that is, when ϕ_1 and ϕ_2 are known, to resolve the pressures in the direction of the axes.

The new quantities are p_1, p_2, and q_3; and the equations are

$$\tan\theta=\frac{q_3}{p_1-p_2}, \qquad (p-q)^2=q_3^{\,2}+(p_1-p_2)^2, \qquad p_1+p_2=p+q$$

It is therefore possible to find the pressures from the curves of equal tint and equal inclination, in any case in which it may be required. In the meantime the curves of figs. 2, 3, 4 shew the correctness of Sir JOHN HERSCHELL's ingenious explanation of the phenomena of heated and unannealed glass.

NOTE A.

As the mathematical laws of compressions and pressures have been very thoroughly investigated, and as they are demonstrated with great elegance in the very complete and elaborate memoir of MM. LAMÉ and CLAPEYRON, I shall state as briefly as possible their results.

Let a solid be subjected to compressions or pressures of any kind, then, if through any point in the solid lines be drawn whose lengths, measured from the given point, are proportional to the compression or pressure at the point resolved in the directions in which the lines are drawn, the extremities of such lines will be in the surface of an ellipsoid, whose centre is the given point.

The properties of the system of compressions or pressures may be deduced from those of the ellipsoid.

There are three diameters having perpendicular ordinates, which are called the *principal axes* of the ellipsoid.

Similarly, there are always three directions in the compressed particle in which there is no tangential action, or tendency of the parts to slide on one another. These directions are called the *principal axes* of *compression* or of *pressure*, and in homogeneous solids they always coincide with each other.

The compression or pressure in any other direction is equal to the sum of the products of the compressions or pressures in the principal axes multiplied into the squares of the cosines of the angles which they respectively make with that direction.

NOTE B.

The fundamental equations of this paper differ from those of NAVIER, POISSON, &c., only in not assuming an invariable ratio between the linear and the cubical elasticity; but since I have not attempted to deduce them from the laws of molecular action, some other reasons must be given for adopting them.

The experiments from which the laws are deduced are—

1*st*, Elastic solids put into motion vibrate isochronously, so that the sound does not vary with the amplitude of the vibrations.

2*d*, REGNAULT's experiments on hollow spheres shew that both linear and cubic compressions are proportional to the pressures.

3*d*, Experiments on the elongation of rods and tubes immersed in water, prove that the elongation, the decrease of diameter, and the increase of volume, are proportional to the tension.

4th, In COULOMB's balance of torsion, the angles of torsion are proportional to the twisting forces.

It would appear from these experiments, that compressions are always proportional to pressures.

Professor STOKES has expressed this by making one of his coefficients depend on the cubical elasticity, while the other is deduced from the displacement of shifting produced by a given tangential force.

M. CAUCHY makes one coefficient depend on the linear compression produced by a force acting in one direction, and the other on the change of volume produced by the same force.

Both of these methods lead to a correct result; but the coefficients of STOKES seem to have more of a real signification than those of CAUCHY; I have therefore adopted those of STOKES, using the symbols m and μ, and the fundamental equations (4.) and (5.), which define them.

NOTE C.

As the coefficient ω, which determines the optical effect of pressure on a substance, varies from one substance to another, and is probably a function of the linear elasticity, a determination of its value in different substances might lead to some explanation of the action of media on light.

V.—*Dissertation on a Peruvian Musical Instrument like the Syrinx of the Ancients.* By Thomas Stewart Traill, M.D., F.R.S.E., Professor of Medical Jurisprudence in the University of Edinburgh.

(Read 1st April 1850.)

The attention which has of late years been paid to the elucidation of the manners and arts of the ancient inhabitants of America, has been productive of the most convincing proofs of the communication between the Eastern and Western Continents at remote but unknown epochs. The learned and highly-interesting researches of **Humboldt** on the antiquities of the New World, have irresistibly led him to this conclusion, which has farther been strengthened by the researches of later travellers. The comparison of the idioms of the Asiatic and American tongues, has hitherto not afforded very direct proof; because the philologist has not yet been put in possession of a sufficient number of materials to make the comparison with advantage. Our ignorance of the languages and customs of Central Asia is a great bar to such studies, and needs not any other illustration than the fact that a highly-polished nation, with a literature and arts hitherto almost unknown in Europe, should have existed for ages in Central Asia. Our countryman, Dr **Gerard**, stimulated by the humane desire of extending the blessings of vaccination to Thibet, has been for some time in that country, and has discovered in its language an Encyclopædia in forty-four volumes, of which the medical part alone fills five volumes; and he finds, that the art of *Lithography*, so new in Europe, has been practised from time immemorial in Kinnaour, a principal city in Thibet, where he found it employed to display the anatomy of the human body. Attempts have been made to supply such deficiencies in the knowledge of Asiatic languages, chiefly by the Germans; especially in the first volume of the *Mithridates* by **Adelung**, and in the *Asia Polyglotta* of **Klaproth**. When our acquaintance with Central Asia shall be more extensive, and the American languages more studied, we may be able to trace the origin of the nations of that continent with greater success; and **Humboldt** does not think it impossible, that traces may yet be discovered in America of tongues and nations that have disappeared from the older hemisphere. It would be curious if future inquirers should discover in America vestiges of those torments of the philologist and antiquary, the Median, Oscan, Phœnician, and Hetruscan tongues.

"If language supply," says **Humboldt**, "but feeble evidence of communication between the two worlds, this communication is fully proved by the cosmogonies, the monuments, the hieroglyphics, and institutions of the people of America and Asia."

It is impossible to consider the Mexican account of the *Serpent-woman, Tonacacihua*, or "Woman of our Flesh," the parent of mankind, with her fall from her state of pristine innocence and happiness; their traditions of a great inundation, in which the human race perished, with the exception of a single family that escaped on a raft; their account of the building of a vast pyramid, which was intended to reach to the sky, and consequent dispersion of the sons of men, and the origin of different languages, caused by the anger of the gods, when they overthrew this monument of human presumption; without perceiving the prototypes of these traditions in the sacred writings of the Hebrews.

The Mexican cosmogony notes *five* epochs of the world; like the people of Thibet, and the Tatar tribes who have retained the ancient religion of the Llama. The first is the age of *earth ;* the second that of *fire ;* the third the age of *wind* or *air ;* the fourth that of *water ; we* live in the *fifth* epoch. It was in the end of the fourth age that the deluge took place, and that a single family was preserved to repeople the earth. As might be expected, *Coxcox* (the Mexican Noah) is represented as the immediate ancestor of the inhabitants of that country. In the first four epochs we may trace the *four ages* of classical antiquity, with the Tatar addition of a fifth. Like the Chinese and Indians, the Mexicans supposed an enormous duration to our earth in all its cataclysms. The Mexican legends extended the age of the world to upwards of 20,000 years.

In the astronomical cycles of some of the American nations, we find strong analogies with the systems of the inhabitants of Thibet, and the various tribes of the Mantscheou Tatars. The Mexican division of the year into 365 days, distributed into 18 months, of 20 days each; the annual intercalation of five days to complete the year, and still more their curious cycle of 52 years, and great cycle of 104 years, in which they intercalated 25 days, to bring the commencement of the next cycle again to correspond with the winter solstice, shew so exact a determination of the true length of the year, that the celebrated LAPLACE is of opinion it could not have originated among a people in so rude a state of society as the Mexicans at their discovery by the Spaniards. The intercalation "of 25 days in 104 years," says he, "supposes a more exact determination of the tropical year than that of HIPPARCHUS, and, what is very remarkable, almost equal to the year of the astronomers of Al-Mamon. When we consider the difficulty of attaining so exact a determination, we are led to believe that it is not the work of the Mexicans, and that it reached them from the Old Continent."

The vast pyramidal temples, accurately placed to the cardinal points, and constructed, as at Cholula, of *sun-dried bricks*, with interposed layers of clay; and occasionally, as at Papantla, with their successive stages neatly covered with hewn stone, sculptured with hieroglyphics, reminded us of the structures of the early ages of Babylonian and Egyptian architecture. The great pyramid of Cholula has a basis 1440 feet on each side, or twice as broad as the great pyramid of Giza; but its height is only 164 English feet. It is built in four stages, and had a small temple on its upper platform, while the interior contained sepulchral chambers;—circumstances which still farther connect this American temple with the pyramids of Egypt, and the Chaldean monuments described by RICH and others. The curious and systematic mode of hieroglyphic paintings of the Mexicans, which combined natural and conventional signs, and, according to HUMBOLDT, also *phonetic* characters, bears a striking similarity to the hieroglyphical papyri of Egypt; and it may not be unworthy of notice, that the Mexican MSS. were folded up *zigzag-wise*, or something like a fan,—precisely almost as the Siamese papyri MSS. are folded to this day.

The singular resemblance between the institutions of the Peruvian lawgiver MANCO CAPAC and the systems of Hindostan, are not to be overlooked; the same exaltation of a theocracy, drawing its descent from heaven; the same exaction of passive obedience to the head of this theocracy, who, like the first legislator of India, traced his pedigree to the sun; the same division of the people into *castes*.

The Peruvians, like the Hindoos, were, by such institutions, trained into a patient, laborious, little-intellectual people; and, like their Asiatic prototypes, have left behind astonishing monuments of patient industry in some of their public works.

I have introduced this comparison between the people of both hemispheres, in order to shew that I do not assume too much in supposing an instrument invented by the ancient inhabitants of the eastern hemisphere, the *original* of the subject of this paper,—*a musical instrument* of stone found in a *Huaca*, or sepulchral tumulus, which is said to have covered the body of an Inca of Peru.

It was brought from South America by my friend JOSHUA RAWDON, Esq. He received it from General PAROISSIEN, a native of England of French extraction, who had obtained it as an article of value and great rarity in Peru.

It was customary with the natives of South America to raise large tumuli over distinguished men; and in these were buried domestic utensils in wood, stone, and the precious metals, often with very considerable treasures, especially in Peru. It would seem that the contents of the rich *Huacas* are still known to the Peruvian Indians, either from tradition or from some species of record. They appear to consider it a sacrilegious act for one of themselves to violate the tomb for the sake of its treasures; but there are more than one instance of their rewarding an European for kindness done them by revealing where he may dig with the certainty of obtaining a golden harvest. The vast Huaca near Truxillo, in the Plain of Chimù, was discovered to JUAN GUTIERRES DE TOLEDO, in 1576, by an Indian, and the bars and utensils of gold it yielded to the fortunate Spaniard equalled 46,810 oz. of gold, or upwards of £181,288 sterling. It appears to have been customary to deposit with the dead the instruments they used, or articles they delighted in; and we may suppose that the Inca with whom this musical instrument was buried was not ignorant of its use. There is no figure of such an instrument among any of the published remains of an American race, as far as my researches have extended; nor am I aware that it has been mentioned among the implements found among them by their Spanish conquerors. It therefore must be of considerably anterior date to the Spanish conquest; as we cannot suppose that since that era, so disastrous to the natives of America, any prince of a native race would have obtained the honours of a *Huaca*, in regions held by the fierce and bigoted conquerors.

Description of the Instrument.

The Peruvian antiquity in question is, in form and principle, similar to the Syrinx of the Greeks and Romans, or Pan's Pipe, well known in England by the somewhat barbarous name of *Pandean Pipes;* and in the Italo-Helvetian cantons by the appropriate denomination of *Organetto,* a diminutive of *Orgaño,* of which it is most probably the prototype.

The Peruvian instrument, however, is not constructed of unequal reeds bound together; but it is cut out of a solid mass of a compact, softish stone, which appears to me to be a variety of Potstone (*Lapis ollaris*). It is cut with great neatness and precision. Its form will be best understood by inspection of the figure. Its sides are not parallel, but they slightly converge toward the upper part of the instrument, for the purpose, apparently, of rendering the orifices of the pieces thin, without endangering the solidity of the whole. The corners of the bottom of the instrument are smoothly and slightly rounded, as if by friction from the hand of the player. The surface seems to have been covered with a brownish shining varnish, similar to the vegetable varnish employed still by the natives on the Essequibo and Orinoco to cover their pottery. It has in part decayed, and in one place bears the impression of cloth of a coarse texture having adhered to it.

The surface, which has evidently been intended for the outside when played, is ornamented with a very regular pattern. The volutes are very neatly executed, and the regular removal of the angular spaces on the right-hand side of the zigzag lines, shews an attempt at variety not unpleasing. The horizontal band of what we would call *Maltese crosses*, is very well executed.

The extreme breadth of the instrument, including the handle, is 6·2 inches; its greatest depth 5·3; the thickness of the body of the instrument is from 0·7 to 0·5 of an inch. The handle projects 1·1 inch from one end, and is perforated by four holes, two of which appear at its extremity, and one on each of its edges, each of them communicating, in the thickness of the handle, with one of the other holes. Their obvious use is to receive a cord, for the convenience of holding the instrument more firmly, or of hanging it up.

There are eight pipes or cylindrical tubes scooped out in the thickness of the stone: they have a diameter of about 0·3 inch, and rise in a sort of general neck three-fourths of an inch above the body of the instrument, forming a horizontal connected series of tubes, which, however, have no communication with each other. Their upper edges, on one side, are slightly thinned, which, no less than the ornaments on the side, shew what part of the tube was pressed against the lips.

These circumstances prove that the Peruvian instrument, like the organetto, was held by the player with the longest tubes, or lowest notes, toward his *right hand.* The depth of the tubes was carefully measured, and is as follows:—

No.		Inches.	No.		Inches.
1.	=	4·90	5.	=	2·45
2.	=	4·50	6.	=	2·85
3.	=	4·12	7.	=	2·00
4.	=	3·50	8.	=	1·58

Though these measurements do not seem quite to accord with the usual proportionate length of pipes with regular musical intervals, they seem to have been adjusted from experimental trials by the maker; and I used every precaution in measuring them with a delicate instrument.

In the common organetto, the tubes are portions of the Spanish reed (*Arundo Donax*), of unequal lengths. These are usually 16 in number; and as each pipe differs from the next a note of the ordinary musical scale, the compass of the instrument, with the usual mode of blowing it, is two octaves. These tubes are open at both ends; and the instrument is tuned by the introduction of a piece of cork, which is pushed farther down when the tone of the note is too sharp, and pushed farther up when the tone is too flat. The key-note is first pitched from some other instrument, or by a tuning-fork; and the other pipes are adjusted by the ear from the key-note.

In the Peruvian instrument the tone of the notes appears to have been adjusted with considerable skill, by careful drilling of the stone; and this has been done by means of a circular drill with cutting edges and a hollowed centre, as the bottom of the holes still shews. The truth of the tones shews that this boring has not been done without repeated trials of the effect; and there is no reason to doubt that the Peruvian artist knew also how to amend the tone by *stopping* the bottom of the pipe when necessary.

The Peruvian instrument has eight notes, in the ordinary way of blowing it; but, by contracting the orifice of the mouth, and by pressing the orifice of the tube toward the lip, an octave to the first is obtained from each note, and if the force of the blast be very strong at the same time, a third octave may be obtained: so that, in the hands of an expert performer, the instrument had considerable compass. In this paper, however, we shall confine our notice to what may be termed the ordinary compass of eight notes, produced by moderate and easy blowing, and producing clear tones.

The Peruvian instrument has a contrivance for giving variety to its notes, which appears to me very ingenious, and which, as far as I can learn, is peculiar to it. Four of its pipes, viz., Nos. 2, 4, 6, and 7, have each a *ventilage*, or small hole perforating its front, about an inch below its top, which must be covered with the fingers of the performer when these pipes are to be sounded. These holes are so near the top of the pipes, that, when open, the sound of the note is quite lost; so that if the performer does not mean to sound that particular note in a rapid movement, it is not necessary to avoid blowing into the pipe, but merely to uncover the ventilage, which effectually destroys its sound. From the peculiar adjustment of the instrument, an harmonious and pleasing *tetrachord* is produced by running up the scale with all the ventilages open.

This description renders it evident that the Peruvian has considerable advantages over the simple Grecian syrinx, which is generally represented in sculpture with *seven* pipes, and occasionally with only *six*. In other respects, the Grecian instrument appears to differ little from the modern organetto; but it is, in some of its modifications, of very high antiquity, and perhaps preceded the invention of the single flute (μοναυλος) with numerous ventilages.

LUCRETIUS describes *Pan's* mode of playing to be the same as we now find it among the Italians:

> " Unco sæpe labro calamos percurrit hianteis
> Fistula sylvestrem ne cesset fundere Musam."
> *Lib.* iv., 592.

The ancients ascribed the invention of the syrinx to the disappointed love of the god PAN, amid the hills of his favourite Arcadia.

> " Pan primus calamos cera conjungere plures
> Instituit." VIRGIL, *Ecl. II.*

Both Pan and the pipe, however, had probably an Egyptian origin, long before the groves of Greece were haunted by any deity; and, if I am not mistaken, we may trace the syrinx to an antediluvian patriarch. *Jubal*, the descendant of CAIN, is in Genesis called "the father of all such as handle the harp and *organ*."

The English translators of the Bible have adopted the interpretation of the Latin Vulgate, in which the Hebrew קובל, *Yogel*,* is rendered *organum*,—" ipse fuit pater canentium cithera et organo." This passage, in the Septuagint, and in the famous Alexandrian MS., runs thus: ὅυτος ἦν ὁ καταδειξας ψαλτηριον καὶ κιθαραν—" He it was who taught the *psaltery* and the harp."†

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* Or, with points, as in WALTON's Polyglott, ר׃קריבל׃.

† The Hebrew name is derived from the verb קבל, which, in the Septuagint, is always rendered by ἐπιτιθημι, *I join together;* which would seem to indicate that it consisted of reeds or pipes *put together*.

What the *Yogel* was has been disputed; but PARKHURST explains it to be a wind instrument of several pipes. The ψαλτηριον of the Septuagint is, by several commentators, said to be a *wind instrument*, or "sort of flute used in churches;" not the modern *psaltery*, which is a trapezoidal flat box, with 13 pairs of strings mounted on two bridges, and played with two crooked sticks.

The invention of the modern organ is a subject of dispute; for few critics will receive S^{ta} CECILIA as the inventor of that noble instrument, although RAFFAELLO has introduced the syrinx in his grand picture of that saint in allusion to this fable. It is of considerable antiquity, however, and it will be sufficient here to remark, that the organ itself is only an adaptation of the more ancient *syrinx* to keys, and an artificial blast of air; and its pipes are tuned on the principle of its venerable prototype.

The ancients seem, however, to have possessed an instrument somewhat in principle resembling the modern organ, in so far as it consisted of several pipes attached to a box, which contained compressed air. In the instrument briefly and obscurely noticed by VITRUVIUS, who lived about the commencement or a little before the Christian era, the air seems to have been compressed by forcing *water* into a brazen box, that communicated with the pipes. The instrument was termed by the inventor, CETESEBIUS of Alexandria, ὕδραυλις; and is attempted to be figured from the description in the Italian translation of VITRUVIUS by BARBATO, Patriarch of Aquileia.

I am indebted to my friend Mr W. CADELL for the notice of a coin of NERO, in the British Museum, on which an 'Οργανον, or perhaps ὕδραυλις, is figured. It seems to be the instrument alluded to by SUETONIUS, in the life of NERO—"reliquam diei partem per *organa hydraulica*, novi et ignoti generis circumduxit." There is a dissertation on the Hydraulis in the Gottingen Transactions.

The organ or psaltery of the book of Genesis, I believe, then, to have been the syrinx; an instrument with which we may reasonably suppose MOSES to have been familiar, as ancient authors generally agree in ascribing the invention of the συριγξ, and of the single flute, μοναυλος, to the Egyptians.

Both flute and syrinx are mentioned by HOMER as known to the Trojans—

'Αυλων Συριγγων τ' ἐνοπην ὁμαδον τ' ἀνθρωπων;

so that, without doubt, the syrinx is an instrument of very great antiquity; and we know that it has been most widely diffused among ancient nations.

Among the Arabs it is in use at the present day. In KŒMPFER's History of Japan, two forms of a syrinx of twelve unequal reeds, used by that people, as also some singular Japanese flutes, are figured in *Tab.* xxxi., A. E. G. J.

From time immemorial, it has been in use among the inhabitants of the Alps; and most of the performers on the *organetto*, who perambulate Europe, bring it from the Italian cantons in the vicinity of the Lake of Como.

That it had been introduced into America, the instrument before you fully proves; and voyagers have discovered a musical instrument very like it in Amsterdam Island, or Tongataboo, in the Pacific Ocean. In a letter which I received from the illustrious **Humboldt** on this subject, he states, that he had found a rude sort of Pan's pipe among the natives on the banks of the Orinoco.—" Il est bien remarquable de vois les mêmes formes se reproduire dans les regions les plus eloignées; j'avais deja été frappé de l'adresse avec laquelle les indigènes de l'Orinoque savoient construire ses *flutes de Pan*, chaque fois que mes canots s'arrêtement là, où le rivage etoit couvert de roseaux."

Scale of the Instrument.

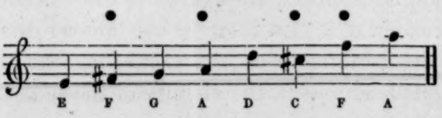
The first attempts at obtaining an idea of the scale of the Peruvian instrument were imperfect, owing to my little skill in either the theory or practice of music. By means of **Broadwood's** C tuning-fork for concert pitch, compared to a piano, I discovered that the lowest note in the Peruvian syrinx was equivalent to E on the first line, and that the next three notes with that formed a tetrachord nearly corresponding to E, F, G, A; that the fifth ascending note was three notes higher than A, or equalled D; that the sixth note was a note *lower* than the preceding; that the seventh was two notes higher than D; and that the eighth was four notes higher than D.

These notes, however, differing from the piano by half a tone, it occurred to me, that, by obtaining the assistance of an accomplished musician on the violincello, the true scale might be ascertained far better than by my unskilful attempts. I employed an expert Italian performer on the organetto to play on the Peruvian instrument, on different evenings, and I was fortunate enough to obtain the assistance of three musical friends, who unite to fine taste great practical skill in music; and to the aid of these gentlemen I am indebted for the following determination of the true scale and powers of the Peruvian syrinx.

The violincello was tuned to the pitch of the Peruvian instrument, and the value of each of its notes was repeatedly tried by this test. The result of these experiments convinced my musical friends, that the maker of that instrument had proceeded on just musical principles in its formation; and that its eight notes were resolvable into two distinct tetrachords, one of which is in a minor, and the other in a major key.

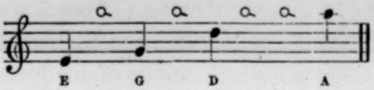
When the ventilages are all *shut*, the following is the

Scale.

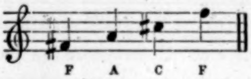


The division of this scale into two tetrachords in different keys is produced by opening the ventilages for the one, and sounding only the notes which were omitted by that process for the other.

Tetrachord in the Key of E *Minor.*



Tetrachord in the Key of F *Major.*



The first tetrachord in the minor key is perfect, and is the most easily performed; for it only requires that all the ventilages be left open, and consequently those notes will not sound. This in all probability was the favourite Peruvian key, and must have imparted, as the minor key always does, a plaintive tone to their music.

The second tetrachord in the major key is nearly perfect: but the instrument on this key is half a note above concert pitch, which throws the F♮ into F♯, and the C♮ into C♯.

It is, however, to be noticed, that, by different modes of ordinary blowing, the tones may be varied nearly half a note; and it is not improbable that the notes now imperfectly pitched, were accurately adjusted by *stuffing*.

The use of the ventilages now becomes very apparent. They enabled the performer to introduce several harmonious modulations, by opening one or more of the holes, without embarrassing him with the attention necessary to avoid the pipes not to be sounded. In this manner considerable variety is given to the succession of sounds, all of which are regulated by the fixed principle of presenting agreeable successions or modulations to the ear. One of my friends was of opinion that some very simple modulations, produced by this means, as an accompaniment to the songs or dances of the Peruvians, was one of the designs of the inventor of the Peruvian syrinx.

The Peruvian performers probably used the succession of simple notes, often reiterated; and we might infer that they often delighted in *slurring* them, by sliding the instrument along the lip, instead of blowing each note distinctly *àllo staccato*, as is usually done by modern performers on the organetto.

It is worthy of remark, that the scale of the Peruvian instrument is founded on a system of *tetrachords*, as was that of a more refined people,—the ancient Greeks. The lyre, according to DIODORUS, was invented by the Egyptian HERMES, and had originally only three strings,—λυϱαν τε ἑυϱειν, ἣν ποιησαι τϱιχοϱδον. The historian says that a fourth, "called μεση, was added by the Muses; that LINUS added the fifth string, named λιχανος; that Orpheus gave it a sixth, ὑπατη; and that the seventh, παρυπατη, was the addition of Thamyris." Even in this improved state of their musical system, the *fourth* was still a favourite and important interval; for we find that their great musical system, as they termed it, "*extended to two octaves composed of five tetrachords*;" in the same manner that the scale of Guido of Arezzo, the inventor of the modern system of musical notation and of *counterpoint*, is composed of different *hexachords*.—See *Burney*.

The sagacity and profound investigations of the learned Sir William Jones have clearly proved that the same systems of literature and arts, which once gave lustre to Ethiopia and Egypt, prevailed in India; and more recent investigations, especially those of Humboldt, Aglio, and several American travellers, have shewn, as we have already noticed, that the arts, the cosmogonies, and astronomy, of the Peruvians, the Mexicans, and some of the other tribes of Central America, betray, in some respects, an Asiatic origin.

VI.—*Some remarks on Theories of Cometary Physics.* By C. Piazzi Smyth, Esq., *F.R.S.E., Astronomer Royal for Scotland, and Professor of Practical Astronomy in the University of Edinburgh.*

(Read 1st April, 1850.)

While the physical appearances of comets have ever excited such intense curiosity and interest, all the theories concerning them are generally confessed to be insufficient to explain them; and, certainly, if we may judge from the various views advocated by different writers, and the anomalous forces gratuitously brought in to support the different hypotheses—it is so.

This unsatisfactory state of things, so different from that in which is the theory of the motions of comets,—seems to be owing partly to the difficulty of making the necessary observations by reason of the undefinable nature of the bodies themselves, and partly from the untoward circumstances under which the observations must be made, as well as the rareness of any opportunities offering. Hence, theories are built upon accounts handed down from old astrological times, when men's prejudices would have prevented them, even if their means had been ample, which they were not, from giving any satisfactory and trustworthy accounts of the phenomena displayed by the heavens of their day.

Then, again, the theories appear to have failed from attempting too much, attempting things not legitimately within their reach; it would have been enough to determine the laws of the changes which the tail undergoes during the orbit of a comet; but in place of this, they attempted to shew why the tails were there, and how they came into existence. This is as much as in the planetary theory to attempt to determine why Saturn has rings; a problem which would have eluded the grasp even of Newton, and will for ever remain wrapped up in the mystery of creation; enough for us that the rings are there; we can measure their diameter and thickness, approximate to their weight, and determine the laws of their rotation, and alternate appearance and disappearance to the earth, and to their own planet; and something of the same sort we may expect to be able to do in the case now before us.

It has been remarked that theory and fact sometimes unite, and that something of theory is necessary to enable us to speak correctly of facts. Many instances of this occur in the history of most of the sciences, but in none have the facts been more misinterpreted by the vulgar feeling of the senses, in the absence of correct theory, than in the case of the physical characteristics of comets. No phenomena were so likely to be misinterpreted by reason of the strong prejudices almost innate in men's minds, as well as the specious and inexplicable character of the appearances themselves. Accordingly, because the tails of comets were seen chiefly at or about the perihelion passages, they were said to be produced then; to have been shot out, and then drawn in again, or dissipated; and numerous have been the theories to explain this creation and extinction. And yet of all the facts that have been ascertained, if any of them can be so considered, with regard to the physical appearances of comets, of none may we be more sure than that the tails of comets, in place of being *largest*, or existing only at the perihelion point of the orbits, are then the *smallest*. Comets of every size, (the distinction of those said to be with and without tails is visionary, or rather the tail is equally a part of the general body of the comet, as the so-called head, and obeys the same laws), when accurately observed, have always been found to decrease in coming to perihelion, and to increase in size in retreating therefrom; this condensation of substance, producing more power to reflect light at that period of the orbit, when, from the closer proximity to the sun, there is more light to reflect. These two causes combining, and both increasing most rapidly with comets of great excentricity and small perihelion distance, occasions the sight, all of a sudden, of a long cometric ray in our skies, when the previous night, or at least the previous clear night, there was none bright enough to catch men's eyes. As the comet leaves the sun, the tail or body expands, and partly from its consequent greater rarity, and the diminishing intensity of its solar illumination, is lost to our sight; and only the denser roundish portion about the head remains visible. This is likewise expanding, and is at length also lost sight of for the same reason. In like manner a comet reappears, first the oval mass about the head, and then the tail gradually strengthens; but its aspect will materially depend not only on its distance from the sun, but on our distance from it, and the direction of our line of sight with the longer axis of the body.

Having had the good fortune to see a rather large number of comets, both great and small, and under circumstances favourable above the average, I hope that I may be of some service to theorists, by stating what data may be looked upon as well fixed with regard to these phenomena; by pointing out some corrections which are absolutely necessary to be made upon the observations, before any good and safe grounds for theorizing can be procured, but which corrections never have been made; and by pointing out the most probable method of improving the observations themselves, which, as at present conducted, are by no means satisfactory.

With regard then to the physical nature of comets, we may take the following as axioms:—

1. A comet consists of a nucleus, and one or more gaseous envelopes.

(1.) No instance has ever been recorded, at least since the fabulous days of astronomy, of a comet having ever been seen without some gaseous appendage, forming, indeed, a distinctive feature at a distance from every other body of the solar system; at a distance, because very close to one of the planets, especially one or two of the asteroids, something of its atmosphere might be observed; but, at the actual distances at which they are viewed, the most powerful telescopes never show these atmospheres in the same manner as those of a comet; they are indicated only by a very different order of phenomena.

When Uranus was first discovered, and no one dreamt of planets beyond Saturn, it was called a comet; not because its form was like that of any recognised comet, but because it was expected that its orbit would prove similar; when, however, the real nature of its path was discovered, the appellation of comet was quickly retracted.

So much for the necessity of a gaseous envelope; of the equal importance of a nucleus, it may be remarked, that although some comets are described as having nuclei, and others as having none; this turns out to be but negative testimony, inasmuch as these latter bodies have always been the fainter, smaller, and more distant ones, in which the nucleus should have been so much the more difficult to distinguish; and if it has not been actually observed itself, there has at least been invariably noticed in every recorded comet, some one point where the gaseous matter was visibly more concentrated than in other parts, indicating thus a virtual or a dark nucleus, if not an actual and a reflective one: while observation, combined with calculation, has satisfactorily shewn, that in comets of every degree of size and excentricity, the mass is so very nearly concentrated in this nucleoid centre, that that need alone be referred to in all determinations of the orbit.

2. The nucleus if solid and material, is exceedingly small.

(2.) Every advance of our knowledge has tended to diminish the possible size of the solid nuclei of comets, planetary perturbation has shewn them to have no sensible mass, and telescopic observation no sensible size; and in the cases of comets of all sizes, observers have witnessed them pass over stars in every position, except, perhaps, exactly centrically with the nucleus, without perceiving any obscuration of the stellar rays.

The old observers have certainly spoken of very large nuclei, but they evidently meant rather the head, which, in some comets at certain parts of the orbit, presents in small telescopes an appearance of planetary opacity and definition.

Such was the case with the great comet of 1843, for three or four days after having passed its perihelion; in small telescopes it was difficult to avoid believing in the existence of an actual planetary nucleus of very notable size; but the fourteen feet reflector of the Cape Observatory shewed the borders of this head to be filmy, and exhibited small stars shining through it; day after day it expanded and became less defined, until at last it ceased to present a solid appearance in any telescope; and at no time was there anything larger than a stellar point, to which the attribute of hard or heavy matter might be expected to apply.

3. The nucleus is excentrically situated in the gaseous body.

(3.) The nucleus actual or virtual, has never been observed in the middle of the envelope, but always nearer one end than the other, the envelopes too, never being round, but invariably more or less elongated.

4. Comets of longest period have the largest bodies.

(4.) This is the general result of cometary statistics, but need not be any more strictly true, than that the largest planets are all at the greatest distances from the sun; they are not strictly ranged in the order of distance agreeably with size, but as a general rule merely, the smaller are closer to the sun than the larger planets. In the same way the telescopic comets, when sufficiently numerous observations have been obtained, have almost always been found to have short periods, and those very brilliant ones, with not only long but broad and dense tails, have invariably been found to be of long period.

5. Those comets whose orbits have the greatest excentricity, are the most excentrically situated in their envelopes, or, vulgarly speaking, have the longest tails.

(5.) This may not be strictly true, but yet is assuredly a very marked feature in the statistics of the question. The great comet of 1843, whose orbit was the most excentric ever known, *i. e.*, had the least perihelion, but great aphelion, distance,—had also the longest and narrowest tail, and the smallest head; consequently the nucleus situated near the centre of the latter was most excentrically situated in the gaseous envelope. HALLEY's comet, and that of 1811, of less excentricity of orbit, had shorter and broader tails and larger heads; and their nuclei, consequently, less excentric: while the telescopic comets of short period, and aphelion not extraordinarily greater than their perihelion distances, exhibit merely somewhat oval masses of vapour.

6. A comet revolves on an axis passing through the nucleus, and at right angles to the major axis of the envelope, in the same period of time that it takes to revolve about the sun: hence, the tail being turned away from the sun in the normal position, is turned away from him in all other parts of the orbit also.

(6.) Every comet has invariably been observed to have its tail turned away from the sun in every part of its orbit; this was the first notable fact established in cometary physics, and the axiom is but a different statement of it.

7. This axis is not at right angles to the plane of the orbit, but variously inclined in the case of different comets, as with the planets.

(7.) There is no reason to expect the contrary; indeed, analogy rather leads us to this conclusion, and it may, if admitted, be sufficient to explain the apparent want of symmetry observed in the tail of HALLEY's comet, that of 1819, and most, if not all, which have been the subject of special attention; and it may tend to account for some of the differences in the appearance of the former body in approaching and leaving its perihelion, at considerable but equal distances on either side of that point.

8. A quicker rotation round the longer axis of the body also appears to exist.

(8.) This seemed to be almost proved by some of the changes which took place in the head of the great comet of 1843, night after night, in the earlier part of its apparition; for instance, a double-winged head, laterally, one night, becoming a single and centrically winged, or rather a tailed-head the next night; but when a body is seen for so very short a space of time, for a few minutes only in twenty-four hours; and sometimes, perhaps, for several days, even that short glimpse is prevented by clouds,—it becomes extremely difficult to separate in such a body as a comet, in which there is nothing decided and tangible, and fixed either in size or brightness, any indications of revolution from those of the other motions and changes which are going on simultaneously. But it seems a point well worthy of attention, and to be proved or disproved.

9. A comet shines by reflected light, and shews a sensible phase; the quantity, form, and position, therefore, of its component matter, cannot be judged of by the eye alone.

(9.) That comets shine by reflected light, is considered to have been proved by ARAGO's polarizing experiment; and was inferred before by every analogy in the planetary system; but all appearance of *phase* has been denied, this, therefore, requires a little explanation. The supposed absence of phase has been attributed to the excessive tenuity of the matter of the comet, and the case has been illustrated by reference to the thin clouds often seen in the west after sunset, or in the east before sunrise, glowing in the solar rays, literally drenched with light, and exhibiting no distinction of light and dark side. A little examination of this instance would have shewn that the conclusion is not so safe; the whole of the cloud being so bright, the difference of illumination of the two sides of it is merely so much the more difficult to distinguish, by reason of the well-known optical or physiological law, that a small difference in the brightness of two objects is more difficult to perceive in proportion to their absolute brightness. If our sensation and means of measurement are not sufficiently accurate in this case of the thin cloud, we have only to turn to a thicker cloud (of the same species, and in a similar part of the sky with regard to the sun), and there we shall see the same law which must obtain in the former case now visibly developed; and then we come to the necessary conclusion, that the illuminated side of every cloud must be brighter than the other, *i. e.*, that it must shew some phase.

The comets are undoubtedly far rarer than any description of cloud floating in our atmosphere, but they are seen under far more favourable circumstances for exhibiting a phase; for, they are illuminated by the sun from one *end*, so that there must be a much greater difference of intensity of light at the two ends than there would be at the two sides, if transversely lighted, as with the long thin films of bright cloud alluded to; and, further, the comet being of the last degree of faintness, the eye is much better able to detect small differences of luminosity. Then again, comets, though they be exceedingly rare, are very voluminous, so that the rays of light have to traverse a great space of matter in passing through them; and if some is reflected in the anterior parts of the body, as we see is the case by the fact that the body is rendered visible to us, there cannot possibly be so strong an illumination on the posterior parts; therefore, we shall either see them fainter than the others; or not at all, if the anterior portions themselves are but just visible.

With these preliminaries then, we may ask, what comet has ever been seen without some phase? for in every single instance, the anterior part of the head, or the denser portion of the envelope, has been brighter than the posterior, exhibiting sometimes the appearance of a luminous sector in front; and the anterior half only, of the body, has been seen, the comet presenting as a general rule two diverging and slightly curved tails. This has been generally held to be merely the effect of looking transversely through a conical envelope of luminous matter, when the ray of light passing through the central portions would meet with less substance, and that part would therefore appear darker than the limbs. This, doubtless, prevails to a great extent, but then we must further remember, that the exterior coats of the envelope will be more strongly illuminated than the interior; and the dark axis of the comet's tail becomes therefore a particular character of phase. Further, as we proceed to the posterior portions of even the outer coats of the envelope, they will be illuminated by a weaker light from the sun, by reason of their greater distance; and if any convergence of them towards the axis should occur, as has actually been observed in some cases, their illuminating rays being then still further diminished in intensity by absorption and reflection, they will hardly be enabled to make themselves visible to us. Thus, the diverging limbs of the tail, and its forked or many-pointed termination, becomes an effect of phase on a body which may be of a symmetrical and rounded, and complete character.

This point, it is of the greatest importance to determine, for if the actual forms of comets be as we see them, they are altogether anomalous in the heavenly regions; and merely on the score of the form of these supposed conical envelopes and diverging streamers, equally anomalous forces have been introduced to explain the phenomena; electricity and polarity, which have no place in any other department of astronomy, being allowed precedence here.

Granting, that a comet is always a prolate spheroidal mass of vapour of different degrees of prolateness, and of actual length in various cases, but always illumined from one end, then we may expect in the larger and denser comets to see but the anterior half of the body; the posterior half being so much further off from the sun, and the rays of light which reach it, being further so much weakened by having passed through the first half; consequently, in this description of comet, we might expect, and we absolutely do find, the phenomenon of the forked tail most marked. In the smaller and fainter comets, on the other hand, the rays of light which reach the posterior half of the body are not much dimmed either from having passed through the excessively tenuous anterior portion, or from having travelled through any notably greater distance from the centre of radiation; in such cases we may expect to see more completely the whole form of the comet; and in them we do actually find nearly, and sometimes quite oval forms, and all gradations from these, through truncated ovals to the forked tails.

These facts induce us to admit the possibility of the bodies of comets being of a far more regular geometric form than has hitherto been suspected, if we allow that conclusions derived from small comets may be safely so extended, *mutatis mutandis*, to large ones; but this view is further confirmed by a notable observation of one of the largest and most excentric of comets.

As already observed, comets decrease in size and increase in density on approaching the perihelion, and the reverse on receding therefrom; hence the phase ought to be most evident, or the tail most forked about the perihelion, both on account of the greater absorption and reflection of light in passing through the denser anterior half, and on account of the greater ratio which the difference of distance of the anterior and posterior ends bear to that of the former and the sun; and at a distance from the perihelion, the effects being reversed, we might expect to see less phase, a less dark axis in the tail, and something of a convergence in the limbs of it. Now, both of these phenomena were distinctly and markedly observable in the great comet of 1843; near the perihelion the tail being forked, the axis almost as dark as the sky round about, and the limbs intensely bright and sharp; but long before it was lost on its retreat to aphelion, the oval darkness was almost obliterated, the whole tail was diffuse, and the posterior portion for fully one-third of the whole visible length shewed a convergence inwards.

But the notable phenomenon is still to come; allowing the above increase of phase in approaching perihelion, it is also evident, that if the perihelion distance be very small, the sun may present a very large angle as viewed from the comet; and in this way rays of light may reach every part of the external coats of the body, and these may be also illumined to that intense degree, that as with the sunrise and sunset clouds already referred to, no phase may be seen; so that, with such comets the maximum of phase will occur a short distance on either side of the perihelion, at and very close to it there will be little or none.

As the comet of 1843 almost touched the sun's surface in passing round it, it must have presented as satisfactory and conclusive a proof as man could have wished for. But although it must have been visible to the naked eye, and to nearly the whole world, in this critical part of its orbit, no mortal man is known to have seen it. Rather a melancholy fact of the imperfection of the astronomical watching of the present age; and it appears all the stronger, from Arago having, in his report to the Academy, descriptive of the discovery of a small comet, enlarged on the perfection of the system of search organized at the Parisian Observatory; by which it appeared that nothing could escape detection; for the assistant who made the discovery, having purposely kept silence when he was relieved in his watch by another person, this one discovered the same comet before having been an hour at his post. But to return to the comet of 1843, it was seen while still not very far from the perihelion, when the sun was still subtending a very large angle, viz., on February 28, the perihelion passage being February 27, 1843; but then only by three persons, or rather parties, and none of them have given sufficiently accurate accounts of what they saw, or have attempted what would have been so invaluable, if effectually and faithfully executed, a *drawing* of the appearances; but their statements, as far as they go, decidedly confirm the views above enumerated. The first of these happy three, with whose account I became acquainted, was a person at the Cape of Good Hope, who (decidedly no scientific person, and having no prejudice in favour of any theory), described the comet as he and his shepherd boy saw it at noonday, a bright hazy star, with the hazy matter streaming off on one side, and *collected into a focus* about two feet *behind it*. Allowing him to have estimated the sun's diameter at one foot, the apparent length of the comet's tail is well given; and the comet itself being spoken of as a bright star in the hazy matter, which streamed off, and collected into a focus at a certain distance behind the head; this certainly may be interpreted into a somewhat symmetrical elliptic figure, having the nucleus in the focus nearest the sun.

The next testimony is from the ship Owen Glendower, the crew and passengers of which ship, when off the Cape on February 28, saw the comet plainly about sunset, "as a short dagger-like object close to the sun." This is not particularly explicit, but yet we may certainly conclude from it, that the comet was broad in the middle of its length, and pointed towards each end, and had little or no axial darkness, which sufficiently conforms with our idea of the perfect shape of the envelope of a comet seen under such circumstances.

The last witness is from the United States, where Mr Clarke, of Portland, saw the comet at 3^h P.M., on the same day, and examined it telescopically, and describes it in these words:—"The nucleus, and also every part of the tail, were as well defined as the moon on a clear day. The nucleus and tail bore the same appearance, and resembled a perfectly pure white cloud without any variation, except a slight change near the head, just sufficient to distinguish the nucleus from the tail at that point." The first sentence well describes the increase of density and definition we have already insisted on as a consequence of so near an approach to the sun; and the second paragraph as perfectly describes the absence of axial darkness, a consequence partly of the increased brightness of the illumination of all the external portion, and partly of its being seen in daylight, and so close to the sun; for then, as every one knows, even the darkest shadows amongst the mountains, and in the craters of the moon, those which appear absolutely black at night, are, under those circumstances, barely distinguishable from the brightest portions. As to the shape, Mr Clarke says,—that the whole body of the comet was seen, but what his idea of the true complete form of its body was, he does not give; but, as he states, that a slight change of brightness near the head was "the only thing to distinguish the nucleus from the tail at that point;" and, further, "that every part of the tail was as well defined as the moon on a clear day;" it would appear to be quite safe to infer, that the tail was not forked; and that the base, instead of a broad or forked, or many-pointed indistinct termination, was as well and sharply outlined as the limbs. A notable distinction this to every subsequent view obtained on succeeding days; and, indeed, in the case of every other comet whatever observed at a great distance from the sun,—when, whatever the definition of the limbs of the tail, the termination or the base has always been so excessively uncertain, that different persons have varied several degrees in assigning the place of it.

Alas! indeed, that the practical astronomy of the present day did not take better account of this unique and critical instance which was offered by the skies of our times; centuries may elapse before another such instance may occur. and this question of the real and complete form of a comet may be in abeyance as long. Something may, doubtless, be done by rigid examination of all the persons who did witness the phenomenon in the comparatively imperfect form of the day after the perihelion passage; but their answers would not be very safe now, so many years after the event, and after the promulgation of a particular theory. Something might also, perhaps, be done, by careful and photometrical observation of the faintest nebulæ, while the darker part of a comet's tail, if it exists, must be passing across them. But this is a very unpromising method, for comets, at all periods, attenuated, become so exceedingly diffuse by the time that they have reached a sufficient distance from the sun, to be viewed for any length of time in a dark sky, and contrasted therein with very faint nebulæ,—that we can hardly expect to obtain any certain indication in this manner. The only sure way is for the comet to be so very close to the sun, that rays from some part or other of his surface will reach every portion of the body of the comet directly, *i. e.*, without having to pass through any other part in order to arrive there.

The fact of this great and invaluable opportunity having been lost, would seem to shew that it is highly desirable that extra meridian observations should be made and watched for by some public observatory in its official routine, instead of being abandoned altogether to amateurs. It is high time that our observatories should be placed in the clearer climates of some of the colonies, and that the most favourable geographical positions should be sought for, rather than the most convenient places in a social point of view; for this results in smoky towns in our own beclouded country being selected as the places where the stars are if possible to be observed.

10. The gaseous envelope is of extreme tenuity, is elastic, and with regard to light is slightly reflective and imperfectly transparent; it decreases in size, but increases in density, and light reflective power in approaching the perihelion, and the reverse when receding from it; and this occurs in a degree proportioned to the excentricity of the orbits of the comets.

(10.) That the gaseous envelope of a comet is of extreme tenuity, and is elastic, slightly reflective and imperfectly transparent, is apparently confessed on all hands, and is proved by the phenomena presented by every comet. That it increases in density and light-reflective power with its proximity to the perihelion, and that this occurs in a degree proportioned to the excentricity of the orbit, requires, at least the latter part does, that the instances on which it is founded should be mentioned; for though the contraction in size of small comets on approaching the sun had been remarked, yet some had maintained it to be accompanied by a decrease in density, by an actual evaporation and disappearance at perihelio; and no one that I am acquainted with had applied it to the larger comets also, or compared the degree of it, with the excentricity of the orbit.

With regard to the effect of excentricity of orbit, a small proportion of it should make a comet visible for a long period on either side of the perihelion, from the lesser degree of attenuation and expansion of its substance at a distance therefrom; and it should also be lost in the sun's rays for a considerable time at and about the perihelion passage, from the matter never being compressed into a sufficiently dense body to be visible in the blaze of day. This rule appears well borne out by both small and large comets; the small ones, for instance Encke's, Biela's, and Faye's, which have for the ratio of the excentricity to the semiaxis major, the numbers respectively, 0·847, 0·755, 0·555, shew no very well marked changes or even characteristics at any part of their orbits, and are soon lost in the twilight even in their densest states; there being little compression, and by axiom 4, little substance to compress, the mean distances being only 2·216, 3·502, and 3·812, the earth's distance being unity.

In the case of Halley's comet, however, the appearances are very different, the excentricity being 0·967, and the semiaxis major 17·988; hence, on this large body approaching the sun and undergoing such a much more intense degree of compression, distinctly marked changes were seen almost from day to day, and at a certain distance from the perihelion it was of great brightness. But the perihelion distance being still large, about half that of the earth, or near fifty millions of miles, the condensation was not sufficient to enable the comet to be seen in moderate twilight, and hence it was not seen after the perihelion passage for more than two months, but then remained visible for nearly four months, so that it was lost sight of at about six months after perihelion passage.

The great comet of 1844–5 had a less perihelion distance, viz., about 25 millions of miles, and a mean distance probably much greater, hence it was sufficiently concentrated in the neighbourhood of the sun to force itself on the notice of men within a week after perihelion: which implies a very much greater degree of brightness, than if Halley's comet had been seen as early, when powerful telescopes, directed by means of an accurate ephemeris, were employed in the search. This comet remained in sight between three and four months, and when last seen was a faint nebulosity with little or no apparent concentration in any part.

But the great comet of 1843 is again the decisive test, as this had a perihelion distance of only half a million miles, 60,000 only from the surface of the sun: here, therefore, we might expect to see the brightness excessive at and about the perihelion; but the subsequent expansion, on account of the great mean distance, would be so rapid that the comet would be soon lost sight of by reason of faintness. Accordingly, we find that this comet pressed itself on men's attention one day only after the perihelion passage; and from its being so very bright then, and yet seen by so few, there can be little doubt but that it might have been observed the day before, if it had been looked for; and would have been so seen, were not staring into the sun's face and immediate vicinity rather a trying, and, consequently, an unpleasant occupation to most eyes, and seldom indulged in, especially in the warmer countries of the south, when the sun might have been that day unveiled from cloud, and was high in the sky. But, however, even the day after the perihelion passage, when the comet must have been much less dense than at that epoch, it was quite bright enough to be seen throughout the day within two degrees of the sun, and was then about one degree in apparent length; four days after it had increased to 25 degrees, in a fortnight to double that; in a month it was so faint and distended as to be lost to most person's eyes, and powerful telescopes only kept it in sight a few days longer. Its meteor-like brightness and short ephemeral existence were subjects of general remark in the south.

This instance may be considered to settle the matter, but Mr Hind's interesting comet of 1847 as a later instance, and a well-marked one also, is very deserving of mention. He discovered this on February 6, 1847, as an exceedingly faint nebulous body approaching perihelion; he observed the gradual condensation in the head and appearance of nucleus and tail, this last being about a degree long on March 9; and having computed the orbit and found the time of perihelion passage to be March 30·269, Gr. M. T., and that the distance from the sun was then only four millions of miles, he called general attention to the circumstance under the hope that, 1*st*, the comet might be seen in daylight on that day; and, 2*d*, that a long tail might be visible in the evening after sunset. In the former he was borne out by the fact, for he observed the comet himself with a refractor of 7 inches aperture at 11^h A.M., within two degrees of the sun, and three other persons are recorded to have witnessed it too. I examined that part of the sky myself on the occasion, but with a telescope of only 3·7 inches aperture could see nothing: Mr Hind himself found it a very difficult object to observe, so that the sizes of the two instruments may be taken as giving some measure of its visibility. In the latter supposition he was not confirmed; for no person saw a tail after sunset, and he himself says that the tail which he saw, exceedingly faint certainly, in the telescope in the day time, very nearly at the epoch of the perihelion passage, was only 40″ long,—but the 90th part of its length 21 days before.

He was led to the first conclusion by the consideration, that the intensity of the light would vary as $\frac{1}{r^2\,\Delta^2}$ (when r is the comet's radius vector, and Δ its true distance from the earth), whence the comet should be at the time of perihelion 230 times brighter than that on March 8, when it was just perceptible to the naked eye. (*Royal Astronomical Society's Monthly Notices*, vol. vii., p. 248.) But here it will be seen that with regard to the distance from the sun and perihelion, the intensity of solar illumination alone is taken account of; but the concentration of the comet at the perihelion must have greatly assisted the effect, and without this it seems pretty certain that the comet would not have been visible : then the introduction of the distance from the earth does not appear correct, for although this may change the apparent diameter of the body, it does not at all alter the intrinsic brightness of the surface.

The second conclusion he was led to, by the old erroneous idea (to use his own words) that "the close approach of the comet to the sun would be likely to *produce* a tail of considerable length :" but in place of so doing it was contracted in size to a very small compass. This additional instance of the prevalence of an idea so completely the reverse of the fact, will, I hope, excuse me from having attempted in so very crude a manner to establish what appears to me the grand statistical truths of cometary physics. But if I have not been able to agree with Mr HIND in his physical ideas, I must express my testimony of his high standing in the more important question of the motions of comets ; here he has indeed filled an honourable niche, which had been long, if not always, unfilled in the cometary credit and fame of this country.

A general result in cometography, certainly following the establishment of this axiom, is, that when the length of the tail of any comet of celebrity is described in millions of miles, a very favourite method with most writers, it will be absolutely necessary to accompany it with an account of the part of its orbit, where the comet is supposed to have been at the time : without this, the statement of an actual length, is as absurd as the fixation of the place of the magnetic pole, without a date being attached.

11. The axis of the tail of a comet is straight at the perihelion, but at any point between this and the aphelion is curved, and is concave toward the latter, the radius of curvature being inversely as the excentricity.

(11.) This I will not attempt to lay much stress upon ; but certainly the tails of the comets of HALLEY, of 1843, and of 1844–5, were sensibly straight near the perihelion ; and the two latter became curved after it, the former more than the latter, and they were concave to the direction in which they were proceeding ; precisely the reverse of the general belief, which states them to bend backwards at the extremity of the tail, as if experiencing some resistance, when whirled round the perihelion with such exceeding velocity.

The direction in which those two comets were proceeding at the time was towards the aphelion ; and I have not had any opportunity of examining a large comet coming up to the perihelion. The great comet of 1843 would have been sufficient to settle the question, but I have only heard of one person (a Commissariat officer voyaging from New South Wales to the Cape), who saw it in the eastern skies before sunrise and the perihelion passage ; and he had made no observations.

12. The molecules composing the envelope of a comet are only held together by their mutual gravitation, each constituting almost a separate projectile, and describing its own parabola about the sun.

The 12th axiom is Sir JOHN HERSCHEL'S, and taken in conjunction with the others, seems generally to explain all the principal variations in appearance, and affords ground for testing each exactly by calculation, and thereby of ascertaining what residual phenomena may be due to laws others than those of gravitation, mechanics, and optics.

After alluding to the observed concentration of ENCKE'S comet near perihelio, and the error of attempting to account for it by the pressure of a supposed æther in the vicinity of the sun, Sir JOHN says (*Royal Astronomical Society's Memoirs*, vol. vi.), "It appears to me that the phenomenon is (if not wholly, at least partially) explicable on a much less gratuitous supposition, viz., that of the extremely feeble attractive force by which the matter of a comet must be held together, owing to the probable minuteness of its mass. Cohesion can hardly be supposed to exist in a gaseous or nebulous body of such tenuity; so that the only bond of union between its molecules must be their feeble gravitation to each other, which is hardly more than mere juxtaposition in space. Hence we must regard each molecule as constituting almost a separate, independent projectile, describing its own parabola about the sun. Now, the interval between two or more parabolas described about a common focus, and having their axes coincident, is a minimum at the perihelion, and increases as we recede from it" in the sesquiplicate ratio of the radius vector. The obervations of ENCKE's comet, which Sir JOHN treated by this theory, shewed rather a more rapid rate of increase and decrease; which might, he thought, be readily accounted for by the effect of the brighter background of sky on which the comet was projected as it approached its perihelion, and *vice versâ*. But whether any other forces may have part in the entire phenomenon presented, he concludes that the property above pointed out, cannot but be allowed to be a *vera causa*, and to have some share in the production of the effect.

To the latter part of this opinion every one must assent; and with respect to the want of agreement between theory and observation, in the case quoted by Sir JOHN of a small comet, in addition to the observations themselves requiring correction, for the cause he has mentioned, the theoretical quantity requires it also on account of the greater attraction of the molecules upon each other at the perihelion, by reason of their increased proximity; while, moreover, the figure of the comet, and the direction in which it is seen, require also to be taken into consideration. With regard to *large* comets, which seem generally to have been thought to be under the dominion of absolutely different laws, the decrease and concentration of the tail at perihelio, is fully accounted for by this 12th axiom, as well as some other phenomena, the perplexing nature of which, when viewed by the light of any other theory, may be gathered by the account given by Sir J. HERSCHEL himself, at the conclusion of the chapter on comets in his work of last year (*Outlines of Astronomy.*)

" It is in a physical point of view that these bodies offer the greatest stimulus to our curiosity. There is, beyond question, some profound secret and mystery concerned in the phenomena of their tails. Perhaps it is not too much to hope that future observations, borrowing every aid from rational speculation, grounded on the progress of physical science generally (especially those branches of it which relate to the ethereal or imponderable elements), may ere long enable us to penetrate this mystery, and to declare whether it is really *matter*, in the ordinary acceptation of the term, which is projected from their heads with such extravagant velocity, and if not impelled, at least *directed*, in its course by a reference to the sun, as its point of avoidance. In no respect is the question as to the materiality of the tail more forcibly pressed on us for consideration, than in that of the enormous sweep that it makes round the sun in perihelio, in the manner of a straight rigid rod, in defiance of the law of gravitation, nay, even of the received laws of motion, extending (as we have seen in the comets of 1680 and 1843)* from near the sun's surface to the earth's orbit, yet whirled round unbroken; in the latter case through an angle of 180° in little more than two hours. It seems utterly incredible that, in such a case, it is one and the same material object which is thus brandished."

This and much more to a similar effect might be quoted from Sir J. HERSCHEL, and other authors, as to the difficulties experienced in the usual method of viewing a comet, as a planetary body at the nucleus, with an appendage attached to, and whirled along with it and by it. No wonder that doubts were expressed as to the attractive force of the small nucleus being able to retain within its grasp portions of matter thrown out to such distances; and that fears were expressed as to the breaking off of the tail; and, because some thought that it ought to bend backwards from the resistance experienced in its course, therefore they said that they did see it bend. Other difficulties also follow from the usual mechanical view of the production of the tail at perihelio, as has been stated by many, from the heat of the sun causing the nucleus to throw out jets of vapour on the side of that luminary, which again has the power to bend them back, and sending them streaming past the nucleus once more, forms the tail. These jets of vapour ought to drive the nucleus away from the sun; for though it may be said that the vapour, being nearly imponderable, should not produce any sensible or visible effect on the nucleus,—yet the nucleus itself is, for anything we know, as imponderable; indeed, if we judge of the masses of the two by the quantity of light reflected, which is almost the only indication we have, and perhaps not a very bad one, then the mass of the tail in most cases exceeds that of the nucleus, *i. e.*, if the quantity of light reflected by the whole envelope were to be concentrated into a single point, it would be brighter than the nucleus. Hence with the extravagant rapidity and the enormous quantity of vapour rushing forth from one side only of the nucleus, that body should be driven far in the opposite direction: but by comparing its observed daily places during the perihelion passages with the computed, we find that no deviations from any such anomalous causes are ever experienced.

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* This is not stated with perfect correctness, at least with regard to the comet of 1843, which might have had a tail of that length some days after the perihelion passage, when it had grown with the rapid increase of its radius vector; but the first day after the perihelion passage, the tail was observed to be only double the sun's diameter (excluding inclination), and its distance must then have been 100 times greater than at the perihelion; so that if the sesquiplicate ratio holds good, we shall have for that epoch a size not very different from planetary bodies. (A curious meeting this must be of the molecules brought for an instant into such close proximity, after having been separated for ages by distances so vast and inconceivable as they must be at the aphelion; and when separating for their diverse orbits, what speculations on their next meeting, not in thunder, lightning, and in rain, but in light and heat unspeakable. On the last occasion, February 1843, the heat was equal (according to Sir J. HERSCHEL) to 47,000 of our suns, 1900 whereof are sufficient to melt the very rocks. Such, at least, must have been the heat, if the comet travelled at that part of its orbit only at the mean rate of the earth; but the velocity was really vastly greater, and the heat much modified thereby. The degree to which velocity in the heavenly spaces may modify distance, in respect of heat, is one still open for inquiry; and the result, in the case of our own earth, as far as it may have been very imperfectly examined into, would lead us to expect that the above proportion would be greatly reduced.)

All these difficulties, however, vanish on considering the envelope of a comet to consist of separate molecules, each constituting an independent projectile, and bound together only by their mutual gravitation and the laws mentioned above; for then the size, character, and position of a comet being given at the perihelion, which we must look on as the normal state, all its principal variations of appearance during the rest of the orbit may be readily computed; and the return of every particle of the envelope to perihelion, or, vulgarly speaking, the retention of the tail by the nucleus, will be no more surprising, nor deviating from ordinary laws, than the return of the nucleus itself: certainly there is nothing "in defiance of the laws of gravitation, and even of the received laws of motion," as stated by a supporter of the other theory to be the case with that.

But following this principle further, we may expect that, in the multitude of molecules moving about amongst each other, occasional conglomerations may occur, after passing the proximity of some large planet, whose attraction acting much more strongly on one part of the envelope than the other, will so much alter the motion of the particles therein, that they will, after some revolutions, gradually collect together at a distance from the nucleus, and at length separate and become a distinct comet. Such a case having actually occurred under our own eyes, four years ago, with BIELA's comet, when it was not at the time under the immediate influence of any planet in particular, nor in any trying part of its orbit round the sun, adds much additional weight to this view of the constitution of such bodies.

This brings us to the second portion of the subject, viz., the corrections which should be applied to apparent observations to deduce the real phenomena.

A comet being an elongated, gaseous, elastic, and semitransparent body, varying in size and density with its distance from the sun, evidently requires many different corrections, according to the point of view and the distance from which it is seen, to give an idea of its real nature at the instant of observation; and needs other corrections, to reduce it from that instant to some other in which it is in a normal condition. This normal condition is plainly in perihelio (though a better general comparison of the volumes of different comets would be obtained by reducing each to its *mean distance*, as they would then be all of much more nearly equal density), and viewed at right angles to the line of its larger axis. This is a position which seems generally to have been taken for granted, though it never occurs even approximately. If we could see a comet at the instant of its perihelio, the plane of its orbit being inclined 90° to that of the ecliptic, and the radius vector being infinitely small, the above view would be nearly obtained, but would gradually be altered as the comet proceeded in its orbit. As seen from the sun, the comet should always present a circular body, and be equally illuminated on all sides, except in so far as the longer axis is inclined to the plane of the orbit: when the comet retreats towards the aphelion, the point of view from the earth becoming more nearly the same as that from the sun,—the comet should become rounder and rounder, as well as larger; and this is found actually to be the case,—the tails of HALLEY's and those of 1843 and 1844-5. having been, at and about the perihelion passage, narrow, and intense, and becoming at the last instant in which they were seen, wide, round, and diffuse.

This, perhaps, together with the facts of phase and imperfect transparency (axioms 9 and 10), is sufficient to shew the importance of correcting the observations for both the terrestrial and the solar elements of effect on the appearance of a comet, and be able to deduce its normal condition. That there may be other changes operating is very probable, but be that as it may, these effects of geometry, mechanics, and optics, actually exist to a very sensible amount, and their corrections must be applied before we can expect to discover any of the residual causes.

I ought, doubtless, to apologize for having formed opinions different from Sir J. HERSCHEL's last, as he is confessedly the person, above all others, entitled to paramount weight in cometary physics; and it may be that I have not properly understood, and unintentionally have not here sufficiently represented the reasons upon which his old conclusions have been altered, and on which he has thought it allowable to introduce electrical and other forces, to explain phenomena amongst the celestial bodies; and I would therefore refer inquirers to his works themselves. But while on the one side, I hope that what has been here brought forward with regard to the complete body of a comet and its symmetrical and geometrical form, when freed of the effects of phase, may remove one of the objections which he felt to allowing the all-sufficiency of gravity acting on a group of independent molecules; on the other side, I not only allow, but think it extremely probable, that some other effects besides those already mentioned, may legitimately occur; and if heat and rotation on the earth produce our trade-winds and hurricanes, much greater effects may follow the more violent alterations of temperature and velocity of motion in a comet. Further, as confirming a curious feature noticed in HALLEY's comet, by Sir JOHN, after the perihelion passage, viz., a long thin ray posterior to the nucleus, to which it might perform the office, he suggested, of conveying back the vapour driven off in front at perihelion; I would mention, that a ray of the same sort was seen posterior to the nucleus of the great comet of 1843, of extravagant length and excessive thinness, appearing as a very fine line of light, and traceable for many degrees up the tail: in these particulars, bearing some relation, perhaps, to the excentricity of the orbit, and to the great length and small breadth of the comet itself. The first day that I saw the comet, three days after perihelio, it was not visible; but, clearly seen on every subsequent day, it existed until the whole was lost in faintness.

The observation of these bodies is not, however, as hinted above, in a sufficiently satisfactory state for rigidly testing any calculable theory. This depends not only on the rarity of the appearance of comets (a matter beyond our control); but also on the insufficient means with which, and the untoward circumstances under which they are generally observed. The telescopes usually turned upon them, have been of so small an optical power, that they would have been considered utterly incompetent for ascertaining the nature of nebulæ high up in the sky; how much more so, when employed on nebulous objects close to the horizon, as the comets usually are, flickering and faint in vapour and smoke, and almost overpowered by the strength of the twilight.

But a sufficiently powerful telescope need not any longer be a difficulty, since the publication of the inventions of the **Earl** of **Rosse**, and Messrs **Lassel** and **Nasmyth**; and the effect of the vapour of the horizon, and the glare of twilight, might be successfully overcome, by establishing an observatory on high land within the tropics, where the geographical position renders the twilight short even to the plains; and the rarity of mountain air would still further reduce the reflective power of the atmosphere. Micrometrical measures, with such instruments, and under such circumstances, should be combined with *photometrical determinations of the brightness of the various parts of the comet*, and *of the background of the sky*. The former observations are easy and straight forward enough, but the latter are difficult and new; the zero must inevitably be taken from a stellar scale, but none such exists at present; for the telescope measure has invariably failed whenever employed for the purpose, and the eye is still thought the best available mean. Hence, none but the brightest stars have had their magnitudes determined, and that but coarsely, while the great question still remains in much the same state as that in which the application of the telescope to divided instruments was in, before men had learnt how to determine the error of collimation. They knew that there were vast powers of accuracy in the optic tubes, but were afraid of great and mysterious errors, which they neither exactly understood, nor saw how to correct. Similarly in photometry, a telescope of large aperture, is confessed to have a larger scale and range than the unassisted eye, but is suspected of misleading to a greater extent.

This is hardly the place for entering into such an experimental branch of practical astronomy; for pointing out what appears to be the error of the methods adopted by others; and for shewing the correctness and efficacy, as I believe, of another plan, which might be adapted to telescopes of any size. But there can be little doubt, from the pressing nature of the demand for advance in this department, that those who are more fortunately situated, must, before long, perfect and employ some method by which the great desideratum for inquiry into cometary physics may be obtained, viz., *drawings*, wherein every feature represented shall be accompanied by numerical statements of its dimensions and brightness, with the *probable error* of each determination. Until this shall have been done, the necessary corrections to reduce the apparent to the real phenomena cannot be undertaken, and I shall therefore hope to return to the subject if it should not be taken up in the meantime by any one better able to conduct it to a successful issue.

VII.—*On the Mechanical Action of Heat, especially in Gases and Vapours.* By William John Macquorn Rankine, Civil Engineer, F.R.S.E., F.R.S.S.A., &c.

(Read 4th February 1850.)

INTRODUCTION.

SUMMARY OF THE PRINCIPLES OF THE HYPOTHESIS OF MOLECULAR VORTICES, AND ITS APPLICATION TO THE THEORY OF TEMPERATURE, ELASTICITY, AND REAL SPECIFIC HEAT.

The ensuing paper forms part of a series of researches respecting the consequences of an hypothesis called that of Molecular Vortices, the object of which is, to deduce the laws of elasticity, and of heat as connected with elasticity, by means of the principles of mechanics, from a physical supposition consistent and connected with the theory which deduces the laws of radiant light and heat from the hypothesis of undulations. Those researches were commenced in 1842, and after having been laid aside for nearly seven years, from the want of experimental data, were resumed in consequence of the appearance of the experiments of M. Regnault on gases and vapours.

The investigation which I have now to describe, relates to the mutual conversion of heat and mechanical power by means of the expansion and contraction of gases and vapours.

In the introduction which I here prefix to it, I purpose to give such a summary of the principles of the hypothesis as is necessary to render the subsequent investigation intelligible.

The fundamental suppositions are the following:—

First,—That each atom of matter consists of a nucleus, or central physical point, enveloped by an elastic atmosphere, which is retained in its position by forces attractive towards the nucleus or centre.

Suppositions similar to this have been brought forward by Franklin, Æpinus, Mossotti, and others. They have in general, however, conceived the atmosphere of each nucleus to be of variable mass. I have treated it, on the contrary, as an essential part of the atom. I have left the question indeterminate, whether the nucleus is a small body of a character distinct from that of the atmosphere, or merely a portion of the atmosphere in a highly condensed state, owing to the mutual attraction of its parts.

According to this first supposition, the boundary between two contiguous atoms of a body is an imaginary surface at which the attractions of all the atomic centres of the body balance each other; and the elasticity of the body is made up of two parts: *First*, the elasticity of the atomic atmospheres at the imaginary boundaries of the atoms, which I shall call the superficial-atomic elasticity; and, *secondly*, the force resulting from the mutual actions of distinct atoms. If the atmospheres are so much condensed round their nuclei or centres, that the superficial-atomic elasticity is insensible, and that the resultants of the mutual actions of all parts of the distinct atoms are forces acting along the lines joining the nuclei or centres, then the body is a perfect solid, having a tendency to preserve not only a certain bulk, but a certain figure; and the elasticity of figure, or *rigidity*, bears certain definite relations to the elasticity of volume.

If the atmospheres are less condensed about their centres, so that the mutual actions of distinct atoms are not reducible to a system of forces acting along the lines joining the atomic centres, but produce merely a cohesive force sufficient to balance the superficial-atomic elasticity, then the condition is that of a *perfect liquid;* and the intermediate conditions between this and perfect solidity constitute the gelatinous, plastic, and viscous states.

When the mutual actions of distinct atoms are very small as compared with the superficial-atomic elasticity, the condition is that of *gas* or *vapour;* and when the substance is so far rarefied that the influence of the atomic nuclei or centres in modifying the superficial elasticity of their atmospheres is insensible, it is then in the state of *perfect gas.*

So far as our experimental knowledge goes, the elasticity of a perfect gas at a given temperature varies simply in proportion to its density. I have therefore assumed this to be the law of the elasticity of the atomic atmospheres, ascribing a specific coefficient of elasticity to each substance.

The *second supposition*, being that from which the hypothesis of molecular vortices derives its name, is the following:—*That the elasticity due to heat arises from the centrifugal force of revolutions or oscillations among the particles of the atomic atmospheres; so that quantity of heat is the* vis viva *of those revolutions or oscillations.*

This supposition appears to have been first definitely stated by Sir HUMPHRY DAVY. It has since been supported by Mr JOULE, whose valuable experiments to establish the convertibility of heat and mechanical power are well known. So far as I am aware, however, its consequences have not hitherto been mathematically developed.

To connect this hypothesis with the undulatory theory of radiation, I have introduced a *third supposition:—That the medium which transmits light and radiant heat consists of the* nuclei *of the atoms, vibrating independently, or almost independently, of their atmospheres;*—so that the absorption of light and of radiant heat, is the transference of motion from the nuclei to their atmospheres, and the emission of light and of radiant heat, the transference of motion from the atmospheres to their nuclei.

Although in all undulations of sensible length and amplitude, such as those of sound, the nuclei must carry their atmospheres along with them, and vibrating thus loaded, produce a comparatively slow velocity of propagation; yet in all probability the minute vibrations of light and radiant heat may be performed by the atomic nuclei in transparent and diathermanous bodies, without moving the atmospheres more than by that amount which constitutes absorption; and those vibrations will therefore be transmitted according to the laws of the elasticity of *perfect solids*, and with a rapidity corresponding to the extreme smallness of the masses set in motion, as compared with the mutual forces exerted by them.

This supposition is peculiar to my own view of the hypothesis, and is, in fact, the converse of the idea hitherto adopted, of an ether surrounding ponderable particles.

The second and third suppositions involve the assumption, that motion can be communicated between the nuclei and their atmospheres, and between the different parts of the atmospheres; so that there is a tendency to produce some permanent condition of motion, which constitutes equilibrium of heat. It is now to be considered what kind of motion is capable of producing increase of elasticity, and what are the conditions of permanency of that motion.

It is obvious, that the parts of the atomic atmospheres may have motions of alternate expansion and contraction, or of rectilinear oscillation about a position of equilibrium, without affecting the superficial atomic elasticity, except by small periodical changes. Should they have motions, however, of *revolution* about centres, so as to form a group of *vortices*, the centrifugal force will have the effect of increasing the density of the atmosphere at what I have called the bounding surfaces of the atoms, and thus of augmenting the elasticity of the body.

In this summary, I shall not enter into the details of mathematical analysis, but shall state results only. The following, then, are the conditions which must be fulfilled, in order that a group of vortices, of small size as compared with the bulk of an atom, and of various diameters, may permanently coexist, whether side by side, or end to end, in the atomic atmospheres of one substance, or of various substances mixed.

First, The mean elasticity must vary continuously; which involves the condition, that at the surface of contact of two vortices of different substances, side by side, or end to end, the respective densities at each point of contact must be inversely proportional to the coefficients of elasticity. Hence *the specific gravities of the* atmospheric parts of *all substances, under precisely similar circumstances as to heat and molecular forces* (a condition realised in perfect gases at the same pressure and temperature), *are inversely proportional to the coefficients of atmospheric elasticity.* Therefore let μ represent the mass of the atmosphere of one atom of any substance, b its coefficient of elasticity, and n the number of atoms which, in the state of perfect gas, occupy unity of volume under unity of pressure at the temperature of melting ice;—then

$$n \mu b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

is a constant quantity for all substances.

Secondly, The superficial elasticity of a vortex must *not* be a function of its diameter: to fulfil which condition, the linear velocity of revolution must be equal throughout all parts of each individual vortex.

Thirdly, In all contiguous vortices of the same substance, the velocities of revolution must be equal; and in contiguous vortices of different substances, the squares of the velocities must be proportional to the coefficients of elasticity of the molecular atmospheres.

The second and third conditions are those of equilibrium of heat, and are equivalent to this law:—

TEMPERATURE *is a function of the square of the velocity of revolution in the molecular vortices divided by the coefficient of elasticity of the atomic atmospheres*;—or

$$\text{Temperature} = \phi \left(\frac{w^2}{b}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

where w represents that velocity.

The *mean* elasticity which a vortex exerts endways is not affected by its motion, being equal to

$$b \varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

where ϱ is its mean density. The superficial elasticity at its lateral surfaces, however, is expressed by

$$\frac{w^2 \varrho}{2g} + b \varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.)}$$

The additional elasticity $\frac{w^2 \varrho}{2g}$, being that which is due to the motion, is independent of the diameter. The divisor g (the force of gravity) is introduced, on the supposition of the density ϱ being measured by weight.

Supposing the atmosphere of an atom to be divided into concentric spherical layers, it may be shewn that the effect of the coexistence of a great number of small vortices in one of those layers whose radius is r, and mean density ϱ, is to give it a centrifugal force, expressed by

$$\frac{w^2}{g r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(V.)}$$

which tends to increase the density and elasticity of the atmosphere at the surface, which I have called the boundary of the atom. The layer is also acted upon by the difference between the mean elasticities at its two surfaces, and by the attraction towards the atomic centre; and these three forces must balance each other.

I have integrated the differential equation which results from this condition, for substances in the gaseous state, in which the forces that interfere with the centrifugal force and atmospheric elasticity are comparatively small; and the result is

$$P = b\frac{\mu}{M}D\left(\frac{w^2}{3gb}+1\right)(1-F)+f(D) \quad . \quad . \quad . \quad \text{(VI.)}$$

P is the entire elasticity of the gas, and D its mean density. M represents the total mass of an atom, measured by weight, and μ that of its atmospheric part; so that $\frac{\mu}{M}D$ is the mean density of the atomic atmospheres.

$f(D)$ denotes the effect of the mutual actions of separate atoms.

The first term represents the superficial-atomic elasticity. F denotes the effect of the attraction of the nucleus in modifying that elasticity, and can be represented approximately by a converging series, in terms of the negative powers of $\frac{w^2}{3gb}+1$, commencing with the inverse square, the coefficients being functions of the density D.

By using the first term of such a series, and determining its coefficient, and the quantity $f(D)$ empirically, I have obtained formulæ agreeing closely with the results of M. REGNAULT's experiments on the Expansion of Atmospheric Air, Carbonic Acid, and Hydrogen.

In a perfect gas, the above expression is reduced to

$$P = b\frac{\mu}{M}D\left(\frac{w^2}{3gb}+1\right) \quad . \quad . \quad . \quad . \quad \text{(VII.)}$$

Let n, as before, denote the number of atoms of a substance which, in the state of perfect gas, occupy unity of volume under unity of pressure at the temperature of melting ice, so that nM is its specific gravity in that state: then

$$P = \frac{D}{nM}n\mu b\left(\frac{w^2}{3gb}+1\right) \quad . \quad . \quad . \quad . \quad \text{(VIII.)}$$

The factor by which $\frac{D}{nM}$ is here multiplied fulfils the condition of being a function of $\frac{w^2}{b}$, and of constants which are the same for all substances, and is therefore fitted for a measure of temperature. It obviously varies proportionally to the pressure of a perfect gas of a given density, or its volume under a given pressure.

Let τ, therefore, denote temperature, as measured from an *imaginary zero*, C degrees of the scale adopted, below the temperature of melting ice, at which

$$\frac{w^2}{3gb}+1=0;$$

Then for all substances

$$\left.\begin{array}{ll} & \tau = C n \mu b \left(\frac{w^2}{3 g b} + 1\right) \\ \text{and in perfect gases} & \\ & P = \frac{\tau}{C} \, \frac{D}{n M} \end{array}\right\} \quad . \ . \ . \quad \text{(IX.)}$$

τ may be termed *absolute temperature*, and the point from which it is measured, the *absolute zero of temperature*. This, as I have observed, is an imaginary point, being lower than the absolute zero of heat by the quantity $C n \mu b$, which is the same for all substances.

The value of C, or the absolute temperature of melting ice, as determined from M. REGNAULT's experiments, is

274°·6 centigrade,

being the reciprocal of

0·00364166 per centigrade degree,

the value to which the coefficients of dilatation of gases at the temperature of melting ice approximate as they are rarefied.

For FAHRENHEIT's scale $C = 494°·28$.

In the sequel I shall represent temperatures measured from that of melting ice by

$$T = \tau - C$$

We have now to consider the absolute quantity of heat, or of molecular *vis viva*, which corresponds to a given temperature in a given substance. It is obvious that

$$\frac{\mu w^2}{2 g}$$

represents, in terms of gravity, the portion of *vis viva*, in one atom, due to the molecular vortices; but besides the vortical motion, there may be oscillations of expansion and contraction, or of rectilinear vibration about a position of equilibrium. The velocity with which these additional motions are performed will be in a permanent condition, when the mean value of its square, independent of small periodic changes, is equal throughout the atomic atmosphere. We may therefore represent by

$$\frac{\mu v^2}{2 g} = k \frac{\mu w^2}{2 g} \quad . \ . \ . \ . \quad \text{(X.)}$$

the total *vis viva* of the atomic atmosphere. To this we have to add that of the nucleus, raising the quantity of heat in one atom to

$$\left.\begin{array}{l} \frac{M v^2}{2 g} = q \quad . \ . \ . \ . \ . \\ \text{While the quantity of heat in unity of weight is} \\ \frac{v^2}{2 g} = Q \quad . \ . \ . \ . \ . \end{array}\right\} \quad \text{(XI.)}$$

The coefficient k (which enters into the value of specific heat) being the ratio of the *vis viva* of the entire motion impressed on the atomic atmospheres by the action of their nuclei, to the *vis viva* of a peculiar kind of motion, may be conjectured to have a specific value for each substance depending in a manner as yet unknown on some circumstance in the constitution of its atoms. Although it varies in some cases for the same substance in the solid, liquid, and gaseous states, there is no experimental evidence that it varies for the same substance in the same condition. In the investigation which follows, therefore, I have treated it as sensibly constant.

The following, then, are the expressions for quantity of heat in terms of temperature. In one atom :—

$$q = \frac{v^2}{2g} \mathrm{M} = \frac{3 k \mathrm{M}}{2 \mathrm{C} n \mu} (\tau - \mathrm{C} n \mu b)$$

In unity of weight :—

$$\mathrm{Q} = \frac{v^2}{2g} = \frac{3 k}{2 \mathrm{C} n \mu} (\tau - \mathrm{C} n \mu b) \tag{XII.}$$

Real specific heat is consequently expressed by the following equations :—

For one atom :—

$$\frac{dq}{d\tau} = \frac{3 k \mathrm{M}}{2 \mathrm{C} n \mu} \quad . \quad . \quad . \quad . \quad .$$

For unity of weight :—

$$\frac{d\mathrm{Q}}{d\tau} = \frac{3 k}{2 \mathrm{C} n \mu} \quad . \quad . \quad . \quad . \quad . \tag{XIII.}$$

For so much of a perfect gas as occupies unity of volume under unity of pressure at the temperature of melting ice :—

$$n \frac{dq}{d\tau} = \frac{3 k \mathrm{M}}{2 \mathrm{C} \mu} \quad . \quad . \quad . \quad . \quad .$$

The laws established experimentally by Dulong, that the specific heats of simple atoms, and of certain groups of compound atoms, bear to each other simple ratios, generally that of equality, and that the specific heats of equal volumes of all simple gases are equal, shew that the specific factor $\frac{k \mathrm{M}}{\mu}$ depends on the chemical constitution of the atom, and thus confirm the conjecture I have stated respecting the coefficient k.

As I shall have occasion, in the investigation which follows, to refer to and to use the equation for the elasticity of vapours in contact with their liquids, which I published in the *Edinburgh New Philosophical Journal for July* 1849, I shall here state generally the nature of the reasoning from which it was deduced.

The equilibrium of a vapour in contact with its liquid depends on three conditions.

First, The total elasticity of the substance in the two states must be the same.

Secondly, The superficial atomic elasticity must vary continuously; so that if at the surface which reflects light there is an abrupt change of density (which seems almost certain), there must there be two densities corresponding to the same superficial-atomic elasticity.

Thirdly, The two forces, which act on each stratum of vapour parallel to the surface of the liquid, namely, the preponderance of molecular attraction towards the liquid, and the difference of the superficial-atomic elasticities at the two sides of the stratum, must be in equilibrio.

Close to the surface of the liquid, therefore, the vapour is highly condensed. The density diminishes rapidly as the distance from the liquid increases, and at all appreciable distances has a sensibly uniform value, which is a function of the temperature and of certain unknown molecular forces.

The integration of a differential equation representing the third condition of equilibrium, indicates the *form* of the approximate equation.

$$\operatorname{Log} P = \alpha - \frac{\beta}{\tau} - \frac{\gamma}{\tau^2} \quad . \quad . \quad . \quad \text{(XIV.)}$$

The coefficients of which have been determined empirically by three experimental data for each fluid. For proofs of the extreme closeness with which the formulæ thus obtained agree with experiment, I refer to the Journal in which they first appeared.

I annex a table of the coefficients for water, alcohol, ether, turpentine, petroleum, and mercury, in the direct equation, and also in the inverse formula,

$$\frac{1}{\tau} = \sqrt{\frac{\alpha - \log P}{\gamma} + \frac{\beta^2}{4\gamma^2}} - \frac{\beta}{2\gamma} \quad . \quad . \quad \text{(XV.)}$$

by which the temperature of vapour at saturation may be calculated from the pressure. In the ninth and tenth columns are stated the limits between which the formulæ have been compared with experiment.

For turpentine, petroleum, and mercury, the formula consists of two terms only,

$$\operatorname{Log} P = \alpha - \frac{\beta}{\tau} \quad . \quad . \quad . \quad \text{(XVI.)}$$

the small range of the experiments rendering the determination of γ impossible.

TABLE *of the Constants in the Formulæ for the Elasticities of Vapours in contact with their Liquids.*

(1.) Names of the Fluids.	(2.) Scale of Pressures.	(3.) Scale of Temperatures.	(4.) a.	(5.) Log. β.	(6.) Log. γ.	(7.) $\frac{\beta}{2\gamma}$	(8.) $\frac{\beta^2}{4\gamma^2}$	(9.) Range of Temperatures.	(10.) Range of Pressures.
Water..................	Millimètres of Mercury.	Centigrade.	7·831247	3·1851091	5·0827176	·0063294	·00004006	Centigrade. −30° to +230°	Millimètres. 0·35 to 20945
Do.	Inches of Mercury.	Fahrenheit.	6·426421	3·4403816	5·5932626	·0035163	·000012364	Fahrenheit. −22° to +446°	Inches. 0·014 to 824·63
Alcohol............ sp. gr. 0·813...	do.	do.	6·16620	3·3165220	5·7602709	·0017998	·000003239	+32° to 264°	0·41 to 165·58
Ether .... boiling at 105° F.	do.	do.	5·33590	3·2084573	5·5119893	·0024856	·000006178	105° to 210°	30·00 to 163·27
Ether ... . boiling at 104° F.	do.	do.	5·44580	3·2571312	5·3962460	·0036296	·000013174	34° to 104°	6·20 to 30·00
Turpentine..........	do.	do.	5·98187	3·5380701	......	......	......	304° to 362°	30·00 to 62·24
Petroleum..........	do.	do.	6·19451	3·5648490	......	......	......	316° to 375°	30·00 to 64·50
Mercury............	Millimètres of Mercury.	Centigrade.	7·5305	3·4685511	......	......	......	Centigrade. 72°·74 to 358°	Millimètres. 0·115 to 760
Do.	Inches of Mercury.	Fahrenheit.	6·1259	3·7238236	......	......	......	Fahrenheit. 162°·93 to 676°·4	Inches. ·0046 to 29·92

The following are some additional values of the constant a for steam, corresponding to various units of pressure used in practice.

Units of Pressure.	Values of a.
ATMOSPHERES of 760 millimètres of mercury, =29·922 inches of mercury, =14·7 lb. on the square inch, =1·0333 kilogrammes on the square centimètre, . .	4·950433
ATMOSPHERES of 30 inches of mercury, =761·99 millimètres, =14·74 lb. on the square inch, =1·036 kilogrammes on the square centimètre, . .	4·949300
Kilogrammes on the square centimètre, ...	4·964658
Kilogrammes on the circular centimètre, ...	4·859748
Pounds avoirdupois on the square inch, ...	6·117662
Pounds avoirdupois on the circular inch, ...	6·012752
Pounds avoirdupois on the square foot, ...	8·276025

All the numerical values of the constants are for common logarithms.

SECTION I.—OF THE MUTUAL CONVERSION OF HEAT AND EXPANSIVE POWER.

(1.) The quantity of heat in a given mass of matter, according to the hypothesis of molecular vortices, as well as every other hypothesis which ascribes the phenomena of heat to motion, is measured by the mechanical power to which that motion is equivalent, that being a quantity the total amount of which in a given system of bodies cannot be altered by their mutual actions, although its distribution and form may be altered. This is expressed in Equation XII. of the Introduction, where the quantity of heat in unity of weight, Q, is represented by the height $\frac{v^2}{2g}$, from which a body must fall in order to acquire the velocity of the molecular oscillations. This height, being multiplied by the weight of a body, gives the mechanical power to which the oscillations constituting its heat are equivalent. The real specific heat of unity of weight, as given in Equation XIII. of the Introduction,

$$\frac{dQ}{d\tau} = \frac{3k}{2Cn\mu}$$

represents the *depth of fall*, which is equivalent to *one degree of rise of temperature* in any given weight of the substance under consideration.

We know, to a greater or less degree of precision, the ratios of the specific heats of many substances to each other, and they are commonly expressed by taking that of water at the temperature of melting ice as unity; but their actual mechanical values have as yet been very imperfectly ascertained, and, in fact, the data necessary for their determination are incomplete.

(2.) Mr JOULE, indeed, has made several very interesting series of experiments, in order to ascertain the quantity of heat developed in various substances by mechanical power employed in different ways, viz., by electric currents excited by the rotation of a magnet, by the forcing of water through narrow tubes, by the agitation of water and oil with a paddle, by the compression of air, and by the friction of air rushing through a narrow orifice. The value of the depth of fall equivalent to a rise of one degree of FAHRENHEIT'S scale in the temperature of a mass of water, as determined by that gentleman, varies, in the different series of experiments, between the limits of 760 feet and 890 feet, the value in which Mr JOULE appears to place the greatest confidence being about 780 feet.

Although the smallness of the differences of temperature measured in those experiments renders the numerical results somewhat uncertain, it appears to me that, as evidence of the convertibility of heat and mechanical power, they are unexceptionable. Nevertheless, there is reason to believe that the true mechanical equivalent of heat is considerably less than any of the values deduced from Mr JOULE'S experiments; for in all of them there are causes of loss of power, the effect of which it is impossible to calculate. In all machinery, a portion of the power which disappears is carried off by waves of condensation and expansion, along the supports of the machine, and through the surrounding air: this portion cannot be estimated, and is, of course, not operative in producing heat within the machine. It is also impossible to calculate, where friction is employed to produce heat, what amount of it has been lost in the production of electricity, a power which is, no doubt, convertible into heat, but which, in such experiments, probably escapes without undergoing that conversion. To make the determination of the mechanical equivalent of heat by electro-magnetic experiments correct, it is necessary that the whole of the mechanical power should be converted into magnetic power, the whole of the magnetic power into what are called electric currents, and the whole of the power of the electric currents into heat, not one of which conditions is likely to be exactly fulfilled. Even in producing heat by the compression of air, it must not be assumed that the whole of the mechanical power is expended in raising the temperature.

(3.) The best means of determining the mechanical equivalent of heat are furnished by those experiments in which no machinery is employed. Of this kind are experiments on the velocity of sound in air and other gases, which, according to the received and well-known theory of LAPLACE, is accelerated by the heat developed by the compression of the medium.

The accuracy of this theory has lately been called in question. There can be no doubt that it deviates from absolute exactness, in so far that the magnitude of the displacements of the particles of air is neglected in comparison with the length of a wave. It appears to me, however, that the Astronomer-Royal, in his remarks on the subject in the *London and Edinburgh Philosophical Magazine for July* 1849, has shewn, in a satisfactory manner, that although the effect of the appreciable magnitude of those displacements, as compared with the length of a wave of sound, is to alter slowly the form of the function representing the wave, still that effect is not sufficiently great to make Laplace's theory practically erroneous. I have, therefore, in the sequel, adhered to the experiments of Dulong, and to those quoted by Poisson, on the velocity of sound, as the best data for determining the mechanical equivalent of heat.

(4.) The expression already given for the real specific heat of unity of weight of a given substance may be resolved into two factors, thus:—

$$\frac{dQ}{d\tau} = \frac{1}{C\,n\,M} \times \frac{3\,k\,M}{2\,\mu} \quad . \quad . \quad . \quad (1.)$$

The first factor, $\frac{1}{C\,n\,M}$, may be considered in general as a known quantity; for C represents, as already stated, 274·6 centigrade degrees, the absolute temperature of melting ice, and $n\,M$ the theoretical weight, in the perfectly gaseous state, of unity of volume of the substance, under unity of pressure, at that temperature; or what is the same thing, $\frac{1}{n\,M}$ is the height of an imaginary column of the substance, of uniform density, and at the temperature of melting ice, whose pressure by weight upon a given area of base is equal to its pressure by elasticity, supposing it to be perfectly gaseous. The determination of the ratio $\frac{3\,k\,M}{2\,\mu}$ is necessary, to complete the solution of the problem.

(5.) The relation now to be investigated between heat and mechanical power, is that which exists between the power expended in compressing a body into a smaller volume, and the increase of heat in consequence of such a compression, and conversely, between the heat which disappears, or, as it is said, becomes *latent*, during the expansion of a body to a greater volume, and the mechanical power gained or developed by that expansion. Those phenomena, according to the hypothesis now under consideration, as well as every hypothesis which ascribes heat to motion, are simply the transformation of mechanical power from one shape into another.

It is obvious, in the first place, without the aid of algebraical symbols, that the general effect of the compression of an oscillating atomic atmosphere, or molecular vortex, must be to accelerate its motion, and of its dilatation, to retard its motion; for every portion of such an atmosphere is urged towards the nucleus or atomic centre by a centripetal force equal to the centrifugal force arising from the oscillation; so that when, by compression, each portion of the atmosphere is made to *approach* the centre by a given distance, the *vis viva* of its motion will be *increased* by the amount corresponding to the centripetal force acting through that distance; and conversely, when by expansion each portion of the atmosphere is made to *retreat* from the centre, the *vis viva* of its motion will be *diminished* by a similar amount.

It is not, however, to be taken for granted, that *all* the power expended in compressing a body appears in the form of heat. More or less power may be consumed or developed by changes of molecular arrangement, or of the internal distribution of the density of the atomic atmospheres; and changes of molecular arrangement or distribution may develope or consume heat, independently of changes of volume.

(6.) We shall now investigate, according to the hypothesis of molecular vortices, the amount of heat produced by an indefinitely small compression of one atom of a body in that state of perfect fluidity which admits of the bounding surface of the atom being treated as if it were spherical: its radius being denoted by R, and the radius of any internal spherical layer of the atmosphere by multiplying R by a fraction u.

I shall denote by the ordinary symbol of differentiation d, such changes as depend on the various positions of portions of the atomic atmosphere relatively to each other, when changes of volume and temperature are not taken into consideration; while by the symbol δ of the calculus of variations, I shall represent such changes as arise from the variations of volume and temperature.

Let us consider the case of an indefinitely thin spherical layer of the atomic atmosphere, whose distance from the nucleus is $R u$, its thickness $R\, d u$, its area $4 \pi R^2 u^2$, and its density $\frac{\mu}{M} D \psi(u, D, \tau)$.

The weight, then, of this layer is

$$4 \pi R^3 \frac{\mu}{M} D u^2 \psi(u, D, \tau)\, d u.$$

Its velocity of oscillation is v, and having, in virtue of that velocity, a mean centrifugal force, as explained in the Introduction (Equation V.), equal to

$$\text{its weight} \times \left(\frac{v^2}{g k R u} = \frac{2 Q}{k R u}\right)$$

it is kept in equilibrio by an equal and opposite centripetal force, arising from attraction and elastic pressure, which is consequently represented by

$$4 \pi R^2 \frac{\mu}{M} \frac{v^2}{g k} D u \psi(u, D, \tau)\, d u$$

$$= 8 \pi R^2 \frac{\mu}{k M} Q D u \psi(u, D, \tau)\, d u.$$

Let the mean density of the atom now be increased by the indefinitely small quantity δD. Then the layer will approach the nucleus through the distance $-\delta(R u) = -u \delta R - R \delta u$, and being acted upon through that distance by the centripetal force already stated, the *vis viva* of oscillation will be increased by a quantity corresponding to the mechanical power (that is to say, the *heat*), represented by the product of that distance by that force, or by

$$-8 \pi R^2 \frac{\mu}{k M} Q D u \psi(u, D, \tau)\, d u \times \delta(R u)$$

$$= -8\pi R^3 \frac{\mu}{k M} Q D \psi(u, D, \tau) u^2 \left(\frac{\delta R}{R} + \frac{\delta u}{u}\right) d u$$

which, because $\frac{\delta R}{R} = -\frac{1}{3} \cdot \frac{\delta D}{D}$, and $\frac{4\pi R^3 D}{3} = M$ is equal to

$$+ Q M \,.\, \frac{2\mu}{k M} \psi(u, D, \tau) u^2 \left(\frac{\delta D}{D} - 3\frac{\delta u}{u}\right) d u.$$

We must suppose that the velocity of oscillation is equalised throughout the atomic atmosphere, by a propagation of motion so rapid as to be practically instantaneous.

Then if the above expression be integrated with respect to $d u$, from $u=0$ to $u=1$, the result will give the whole increase of heat in the atom arising from the condensation δD; and dividing that integral by the atomic weight M, we shall obtain the corresponding development of heat in unity of weight. This is expressed by the following equation :—

$$\delta Q' = 2 Q \frac{\mu}{k M} \left\{ \frac{\delta D}{D} \int_0^1 d u \,.\, u^2 \psi(u, D, \tau) - 3 \int_0^1 d u \,.\, u \,\delta u \,\psi(u, D, \tau) \right\} \quad . \quad . \quad (2.)$$

The letter Q′ is here introduced to denote, when negative, that heat which is consumed in producing changes of volume and of molecular arrangement, and when positive, as in the above equation, the heat which is produced by such changes.

The following substitutions have to be made in Equation (1.) of this Section.

For Q is to be substituted its value, according to Equation XII. of the Introduction; or abbreviating $C n \mu b$ into κ:—

$$Q = \frac{3 k}{2 C n \mu} (\tau - \kappa) \quad . \quad . \quad . \quad (3.)$$

The value of the first integral in Equation (2.) of this Section is

$$\int_0^1 d u \,.\, u^2 \psi(u, D, \tau) = \frac{1}{3}$$

The value of the second integral

$$-3 \int_0^1 d u \,.\, u \,\delta u \,\psi(u, D, \tau)$$

remains to be investigated. The first step in this inquiry is given by the condition, that whatsoever changes of magnitude a given spherical layer undergoes, the portion of atmosphere between it and the nucleus is invariable. This condition is expressed by the equation

$$0 = \left(\delta u \frac{d}{d u} + \delta \tau \frac{d}{d \tau} + \delta D \frac{d}{d D}\right) \int_0^u d u \,.\, u^2 \psi(u, D, \tau) \quad . \quad . \quad . \quad (4.)$$

from which it follows that

$$\delta u = -\frac{1}{u^2 \psi(u, D, \tau)} \left(\delta \tau \frac{d}{d \tau} + \delta D \frac{d}{d D}\right) \int_0^u d u \,.\, u^2 \psi(u, D, \tau)$$

and consequently that

$$-3\int_0^1 du\,.\,u\,\delta u\,\psi(u,\mathrm{D},\tau)=$$
$$+\left(\delta\tau\,.\,\frac{d}{d\tau}+\delta\mathrm{D}\frac{d}{d\mathrm{D}}\right)3\int_0^1\frac{du}{u}\int_0^u du\,.\,u^2\psi(u,\mathrm{D},\tau)$$

Hence, making

$$9\int_0^1\frac{du}{u}\int_0^u du\,.\,u^2\psi(u,\mathrm{D},\tau)=\mathrm{U}\quad .\quad .\quad .\quad (5.)$$

The second integral in Equation (2.) is transformed into

$$+\frac{1}{3}\left(\delta\tau\frac{d}{d\tau}+\delta\mathrm{D}\frac{d}{d\mathrm{D}}\right)\mathrm{U}.$$

By means of those substitutions we obtain, for the mechanical value of the heat developed in unity of weight of a fluid by any indefinitely small change of volume or of molecular distribution—

$$\delta Q=\frac{\tau-\kappa}{\mathrm{C}n\mathrm{M}}\left(\delta\mathrm{D}\left(\frac{1}{\mathrm{D}}+\frac{d\mathrm{U}}{d\mathrm{D}}\right)+\delta\tau\frac{d\mathrm{U}}{d\tau}\right)$$

or taking $\mathrm{V}=\frac{1}{\mathrm{D}}$ to denote the volume of unity of weight of the substance, (6.)

$$\delta Q=-\frac{\tau-\kappa}{\mathrm{C}n\mathrm{M}}\left(\delta\mathrm{V}\left(\frac{1}{\mathrm{V}}-\frac{d\mathrm{U}}{d\mathrm{V}}\right)-\delta\tau\frac{d\mathrm{U}}{d\tau}\right)$$

Of this expression, the portion $\frac{\tau-\kappa}{\mathrm{C}n\mathrm{M}}\cdot\frac{\delta\mathrm{D}}{\mathrm{D}}=-\frac{\tau-\kappa}{\mathrm{C}n\mathrm{M}}\cdot\frac{\delta\mathrm{V}}{\mathrm{V}}$ represents the variation of heat arising from mere change of volume.

$\frac{\tau-\kappa}{\mathrm{C}n\mathrm{M}}\delta\mathrm{V}\frac{d\mathrm{U}}{d\mathrm{V}}=\frac{\tau-\kappa}{\mathrm{C}n\mathrm{M}}\delta\mathrm{D}\frac{d\mathrm{U}}{d\mathrm{D}}$, denotes the variation of heat produced by change of molecular distribution dependent on change of volume.

$\frac{\tau-\kappa}{\mathrm{C}n\mathrm{M}}\delta\tau\frac{d\mathrm{U}}{d\tau}$ expresses the variation of heat due to change of molecular distribution dependent on change of temperature.

(7.) The function U is one depending on molecular forces, the nature of which is as yet unknown. The only case in which it can be calculated directly is that of a perfect gas. Without giving the details of the integration, it may be sufficient to state, that in this case

$$\mathrm{U}=\frac{\kappa}{\tau}$$

and therefore that . . . (7.)

$$\frac{d\mathrm{U}}{d\tau}=-\frac{\kappa}{\tau^2};\qquad\frac{d\mathrm{U}}{d\mathrm{V}}=0$$

In all other cases, however, the value of this function can be determined indirectly, by introducing into the investigation the principle of the *conservation of vis viva*.

Suppose a portion of any substance, of the weight *unity*, to pass through a variety of changes of temperature and volume, and at length to be brought back to its primitive volume and temperature. Then the absolute quantity of heat in the substance, and the molecular arrangement and distribution, being the same as at first, the effect of their changes is eliminated; *and the algebraical sum of the vis viva expended and produced, whether in the shape of expansion and compression, or in that of heat, must be equal to zero*:—that is to say, if, on the whole, any mechanical power has appeared, and been given out from the body, in the form of expansion, an equal amount must have been communicated to the body, and must have disappeared in the form of heat; and if any mechanical power has appeared and been given out from the body in the form of heat, an equal amount must have been communicated to the body, and must have disappeared in the form of compression. This principle expressed symbolically is

$$\Delta \Pi + \Delta Q' = 0 \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Where Π, when positive, represents expansive power given out, when negative, compressive power absorbed; and Q' represents, when positive, heat given out, when negative, heat absorbed.

To take the simplest case possible, let the changes of temperature and of volume be supposed to be indefinitely small, and to occur during distinct intervals of time, so that τ and V are independent variables. Let the initial absolute temperature be τ, the initial volume V, and the initial total elasticity P; and let the substance go through the following four changes.

First, Let its temperature be raised from τ to $\tau + \delta\tau$, the volume remaining unchanged. Then the quantity of heat absorbed is

$$-\delta\tau \left(\frac{dQ}{d\tau} - \frac{\tau - \kappa}{C\,n\,M}\,\frac{dU}{d\tau}\right)$$

and there is no expansion nor compression.

Secondly, Let the body expand, without change of temperature, from the volume V to the volume $V + \delta V$. Then the quantity of heat absorbed is

$$-\delta V \,.\, \frac{\tau + \delta\tau - \kappa}{C\,n\,M}\left(\frac{1}{V} - \frac{d}{dV}\,.\,\left(U + \frac{dU}{d\tau}\,\delta\tau\right)\right)$$

while the power given out by expansion is

$$\delta V \left(P + \frac{dP}{d\tau}\,\delta\tau\right)$$

Thirdly, Let the temperature fall from $\tau + \delta\tau$ to its original value τ, the volume $V + \delta V$ continuing unchanged; then the heat given out is

$$+\delta\tau \left(\frac{dQ}{d\tau} - \frac{\tau - \kappa}{C\,n\,M}\,\frac{d}{d\tau}\left(U + \frac{dU}{dV}\,\delta V\right)\right)$$

and there is no expansion nor compression.

Fourthly, Let the body be compressed, without change of temperature, to its original volume V; then the heat given out is

$$+\delta \mathrm{V} \frac{\tau-\kappa}{\mathrm{C}\, n\, \mathrm{M}}\left(\frac{1}{\mathrm{V}}-\frac{d\,\mathrm{U}}{d\,\mathrm{V}}\right)$$

while the power absorbed in compression is

$$-\delta \mathrm{V} \,.\, \mathrm{P}$$

The body being now restored in all respects to its primitive state, the sum of the two portions of power connected with change of volume, must, in virtue of the principle of *vis viva,* be equal to the sum of the four quantities of heat with their signs reversed. Those additions being made, and the sums divided by the common factor $\delta \mathrm{V}\, \delta \tau$, the following equation is obtained:—

$$\frac{d\,\mathrm{P}}{d\,\tau}=\frac{1}{\mathrm{C}\, n\, \mathrm{M}}\left(\frac{1}{\mathrm{V}}-\frac{d\,\mathrm{U}}{d\,\mathrm{V}}\right) \quad . \quad . \quad . \quad (9.)$$

The integral of this partial differential equation is

$$\mathrm{U}=\phi \,.\, \tau+\int d\,\mathrm{V}\left(\frac{1}{\mathrm{V}}-\mathrm{C}\, n\, \mathrm{M} \frac{d\,\mathrm{P}}{d\,\tau}\right) \quad . \quad . \quad . \quad (10.)$$

Now $\phi \,.\, \tau$ being the same for all densities, is the value of U for the perfectly gaseous state, or $\frac{\kappa}{\tau}$; for in that state, the integral $= 0$.

The values of the partial differential coefficients are accordingly—

$$\left.\begin{aligned} \frac{d\,\mathrm{U}}{d\,\mathrm{V}}&=\frac{1}{\mathrm{V}}-\mathrm{C}\, n\, \mathrm{M} \frac{d\,\mathrm{P}}{d\,\tau} \\ \frac{d\,\mathrm{U}}{d\,\tau}&=-\frac{\kappa}{\tau^2}-\mathrm{C}\, n\, \mathrm{M} \int d\,\mathrm{V} \,.\, \frac{d^2\,\mathrm{P}}{d\,\tau^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (11.)$$

and they can, therefore, be determined in all cases in which the quantity $\kappa=\mathrm{C}\, n\, \mu\, b$, and the law of variation of the total elasticity with the volume and temperature are known, so as to complete the data required in order to apply equation 6 of this section to the calculation of the mechanical value of the variations of heat due to changes of volume and molecular arrangement.

The total elasticity of an imperfect gas, according to Equations VI. and XII. of the introduction, being

$$\mathrm{P}=\frac{\tau}{\mathrm{C}\, n\, \mathrm{M}\, \mathrm{V}}\left(1-\mathrm{F}\left(\mathrm{D}, \frac{\tau}{\kappa}\right)\right)+f(\mathrm{D})$$

its first and second partial differential coefficients with respect to the temperature are,—

$$\frac{d\,\mathrm{P}}{d\,\tau}=\frac{1}{\mathrm{C}\, n\, \mathrm{M}\, \mathrm{V}}\left(1-\left(1+\tau \frac{d}{d\,\tau}\right) \mathrm{F}\left(\mathrm{D}, \frac{\tau}{\kappa}\right)\right)$$

$$\frac{d^2\,\mathrm{P}}{d\,\tau^2}=-\frac{1}{\mathrm{C}\, n\, \mathrm{M}\, \mathrm{V}}\left(2 \frac{d}{d\,\tau}+\tau \frac{d^2}{d\,\tau^2}\right) \mathrm{F}\left(\mathrm{D}, \frac{\tau}{\kappa}\right)$$

Consequently, for the imperfectly gaseous state,

$$\left.\begin{aligned} \mathrm{U} &= \frac{\kappa}{\tau} + \left(1 + \tau \frac{d}{d\tau}\right) \int d\mathrm{V} \, . \, \frac{\mathrm{F}\left(\mathrm{D}, \frac{\tau}{\kappa}\right)}{\mathrm{V}} \\ \frac{d\mathrm{U}}{d\mathrm{V}} &= \frac{1}{\mathrm{V}} \cdot \left(1 + \tau \frac{d}{d\tau}\right) \mathrm{F}\left(\mathrm{D}, \frac{\tau}{\kappa}\right) \\ \frac{d\mathrm{U}}{d\tau} &= - \frac{\kappa}{\tau^2} + \left(2 \frac{d}{d\tau} + \tau \frac{d^2}{d\tau^2}\right) \int d\mathrm{V} \, . \, \frac{\mathrm{F}\left(\mathrm{D}, \frac{\tau}{\kappa}\right)}{\mathrm{V}} \end{aligned}\right\} \quad . \quad . \quad (12.)$$

(8.) It is to be observed that the process followed in ascertaining the nature of the function U is analogous to that employed by M. CARNÔT in his theory of the motive power of heat, although founded on contrary principles, and leading to different results.

CARNÔT, in fact, considers heat to be something of a peculiar kind, whether a condition or a substance, the total amount of which in nature is incapable of increase or of diminution. It is not, therefore, according to his theory, convertible into mechanical power; but is capable, by its transmission through substances under particular circumstances, of causing mechanical power to be developed. He supposes a body to go through certain changes of temperature and volume, and to return at last to its primitive volume and temperature, and conceives, in accordance with his view of the nature of heat, that it must have given out exactly the same quantity of heat that it has absorbed. The transmission of this heat he regards as the cause of the production of an amount of mechanical power, depending on the quantity of heat transmitted and on the temperature at which the transmission has taken place. According to these principles, a body, having received a certain quantity of heat, is capable of giving out not only all the heat it has received, but also a quantity of mechanical power which did not before exist.

According to the theory of this Essay, on the contrary, and to every conceivable theory which regards heat as a modification of motion, no mechanical power can be given out in the shape of expansion unless the quantity of heat emitted by the body in returning to its primitive temperature and volume is *less* than the quantity of heat originally received: the excess of the latter quantity above the former disappearing as heat, to appear as expansive power, so that the sum of the *vis viva* in those two forms continues unchanged.

SECTION II.—OF REAL AND APPARENT SPECIFIC HEAT, ESPECIALLY IN THE STATE OF PERFECT GAS.

(9.) The apparent specific heat of a given substance is found by adding to the real specific heat (or the heat which retains its form in producing an elevation of one degree of temperature in unity of weight) that additional heat which disappears in producing changes of volume and of molecular arrangement, and which is determined by reversing the sign of Q^1 in equation 6 of Section I. (so as to transform it from heat evolved to heat absorbed), and taking its *total* differential coefficient with respect to the temperature. Hence, denoting total apparent specific heat by K,—

$$K = \frac{dQ}{d\tau} - \frac{d.Q^1}{d\tau} = \frac{dQ}{d\tau} - \frac{dQ^1}{d\tau} - \frac{dQ^1}{dV} \cdot \frac{dV}{d\tau}$$

$$= \frac{1}{CnM} \left\{ \frac{3kM}{2\mu} + (\tau - \kappa) \left(\frac{dV}{d\tau} \left(\frac{1}{V} - \frac{dU}{dV} \right) - \frac{dU}{d\tau} \right) \right\} \quad (13.)$$

Another mode of expressing this coefficient is the following:—

$$\left. \begin{array}{ll} \text{Denote the ratio} & \dfrac{2\mu}{3kM} \text{ by } N, \\ \text{and the real specific heat by } \mathfrak{k} & \\ & = \dfrac{1}{CnMN} \end{array} \right\} \quad . \quad . \quad (14.)$$

Then

$$K = \mathfrak{k} \left\{ 1 + N(\tau - \kappa) \left(\frac{dV}{d\tau} \left(\frac{1}{V} - \frac{dU}{dV} \right) - \frac{dU}{d\tau} \right) \right\} \quad (15.)$$

The value of $\frac{dV}{d\tau}$ is to be determined from the conditions of each particular case; so that each substance may have a variety of apparent specific heats, according to the manner in which the volume varies with the temperature.

If the volume is not permitted to vary, so that $\frac{dV}{d\tau} = 0$, there is obtained the following result, being the apparent specific heat at constant volume:—

$$K_v = \frac{1}{CnM} \left(\frac{1}{N} - (\tau - \kappa) \frac{dU}{d\tau} \right)$$

$$= \mathfrak{k} \left(1 - N(\tau - \kappa) \frac{dU}{d\tau} \right) \quad . \quad . \quad (16.)$$

(10.) When the substance under consideration is a perfect gas, it has already been stated (Eq. 7), that $\frac{dU}{d\tau} = -\frac{\kappa}{\tau^2}, \frac{dU}{dV} = 0$; and because the volume of unity of weight is directly as the absolute temperature and inversely as the pressure,

$$\frac{1}{V}\frac{dV}{d\tau} = \frac{1}{\tau} - \frac{1}{P}\frac{dP}{d\tau} \quad . \quad . \quad . \quad (17.)$$

Hence the following are the values of the apparent specific heats of unity of weight of a theoretically perfect gas under different circumstances:—

$$\left.\begin{array}{l}
\text{General value of the total apparent specific heat:—} \\
\mathrm{K}=\dfrac{1}{\mathrm{C}\,n\,\mathrm{M}}\left\{\dfrac{1}{\mathrm{N}}+(\tau-\kappa)\left(\dfrac{\kappa}{\tau^2}+\dfrac{d\mathrm{V}}{\mathrm{V}\,d\tau}\right)\right\} \\
\quad=\dfrac{1}{\mathrm{C}\,n\,\mathrm{M}}\left\{\dfrac{1}{\mathrm{N}}+(\tau-\kappa)\left(\dfrac{\kappa}{\tau^2}+\dfrac{1}{\tau}-\dfrac{d\mathrm{P}}{\mathrm{P}\,d\tau}\right)\right\} \\
\text{Apparent specific heat at constant volume:—} \\
\mathrm{K_v}=\dfrac{1}{\mathrm{C}\,n\,\mathrm{M}}\left\{\dfrac{1}{\mathrm{N}}+\dfrac{\kappa}{\tau}-\dfrac{\kappa^2}{\tau^2}\right\} \\
\quad=\mathfrak{k}\left(1+\mathrm{N}\left(\dfrac{\kappa}{\tau}-\dfrac{\kappa^2}{\tau^2}\right)\right) \\
\text{Apparent specific heat under constant pressure:—} \\
\mathrm{K_P}=\dfrac{1}{\mathrm{C}\,n\,\mathrm{M}}\left(\dfrac{1}{\mathrm{N}}+1-\dfrac{\kappa^2}{\tau^2}\right) \\
\quad=\mathfrak{k}\left\{1+\mathrm{N}\left(1-\dfrac{\kappa^2}{\tau^2}\right)\right\}
\end{array}\right\}\quad .\ .\ .\quad (18.)$$

The ratio of the apparent specific heat under constant pressure to the apparent specific heat at constant volume is the following:—

$$\frac{\mathrm{K_P}}{\mathrm{K_v}}=\frac{1+\mathrm{N}\left(1-\dfrac{\kappa^2}{\tau^2}\right)}{1+\mathrm{N}\left(\dfrac{\kappa}{\tau}-\dfrac{\kappa^2}{\tau^2}\right)}=1+\mathrm{N}\,\frac{1-\dfrac{\kappa}{\tau}}{1+\mathrm{N}\left(\dfrac{\kappa}{\tau}-\dfrac{\kappa^2}{\tau^2}\right)}\quad .\ .\ .\quad (19.)$$

The value of κ is unknown; and, as yet, no experimental data exist from which it can be determined. I have found, however, that practically, results of sufficient accuracy are obtained by regarding κ as so small in comparison with τ, that $\frac{\kappa}{\tau}$, and *à fortiori* $\frac{\kappa^2}{\tau^2}$, may be neglected in calculation.

Thus are obtained the following *approximate* results, for perfect gases, and gases which may without material error be treated as perfect.

$$\left.\begin{array}{l}
\text{General value of the total apparent specific heat:—} \\
\mathrm{K}=\dfrac{1}{\mathrm{C}\,n\,\mathrm{M}}\left(\dfrac{1}{\mathrm{N}}+\dfrac{\tau}{\mathrm{V}}\dfrac{d\mathrm{V}}{d\tau}\right)=\mathfrak{k}+\mathrm{P}\dfrac{d\mathrm{V}}{d\tau} \\
\quad=\dfrac{1}{\mathrm{C}\,n\,\mathrm{M}}\left(\dfrac{1}{\mathrm{N}}+1-\dfrac{\tau}{\mathrm{P}}\dfrac{d\mathrm{P}}{d\tau}\right) \\
\text{Apparent specific heat at constant volume:—} \\
\mathrm{K_v}=\dfrac{1}{\mathrm{C}\,n\,\mathrm{M}\,\mathrm{N}}=\mathfrak{k} \\
\text{being equal to the real specific heat.} \\
\text{Apparent specific heat under constant pressure:—} \\
\mathrm{K_P}=\dfrac{1}{\mathrm{C}\,n\,\mathrm{M}}\left(\dfrac{1}{\mathrm{N}}+1\right)=\mathfrak{k}\,(1+\mathrm{N})
\end{array}\right\}\quad .\ .\quad (20.)$$

Ratio of those two specific heats:—

$$\frac{K_p}{K_v} = 1 + N \quad . \quad . \quad . \quad . \quad (21.)$$

This ratio is the quantity called by **Poisson** γ, in his researches on the propagation of sound.

(11.) It is unnecessary to do more than to refer to the researches of **Poisson**, and to those of **Laplace**, for the proof that the effect of the production of heat by the compression of air is the same as if the elasticity varied in proportion to that power of the density whose index is the ratio of the two specific heats; so that the actual velocity of sound is greater than that which it would have if there were no such development of heat, in the proportion of the square root of that ratio.

The following is the value of the velocity of sound in a gas, as given by **Poisson**, in the second volume of his *Traité de Mécanique*:—

$$a = \sqrt{g \cdot \gamma \cdot (1 + E\,T)\frac{m\,h}{\Delta}} \quad . \quad . \quad . \quad . \quad (22.)$$

where a denotes the velocity of sound, g the velocity generated by gravity in unity of time, E the coefficient of increase of elasticity with temperature, at the freezing point of water, T the temperature measured from that point, m the specific gravity of mercury, Δ that of the gas at the temperature of melting ice, and pressure corresponding to a column of mercury of the height h. It follows that the ratio γ is given by the formula

$$\gamma = 1 + N \text{ nearly} = \frac{a^2\,\Delta}{g\,m\,h\,(1 + E\,T)} \quad . \quad . \quad . \quad (23.)$$

Calculations have been made to determine the ratio γ from the velocity of sound; but as many of them involve erroneous values of the coefficient of elasticity E, the experiments have to be reduced anew.

The following calculation is founded on an experiment quoted by **Poisson** on the velocity of sound in atmospheric air, the values of E, m, and Δ being taken from the experiments of M. **Regnault**.

a = 340·89 mètres per second.

$g = 9^{m}{\cdot}80896$. $h = 0^{m}{\cdot}76$. T = 15°·9 Centigrade.

$$E = 0{\cdot}003665; \quad \frac{m}{\Delta} = 10513.$$

Consequently, for atmospheric air,

$$\gamma = 1{\cdot}401.$$

The results of a reduction, according to correct data, of the experiments of **Dulong** upon the velocity of sound in atmospheric air, oxygen, and hydrogen, are as follows:—

Atmospheric air, ...	γ = 1·410
Oxygen, ...	1·426
Hydrogen, ...	1·426

Thus it appears, that for the simple substances, oxygen and hydrogen, the ratio N is the same, while for atmospheric air it is somewhat smaller.*

(12.) The ordinary mode of expressing the specific heats of gases is to state their ratios to that of an *equal volume* of atmospheric air at the same pressure and temperature.

When $\frac{\kappa}{\tau}$ is a very small fraction, specific heats of *unity of volume* of a perfect gas are given by the equations

$$\left.\begin{aligned} n\,\mathrm{M}\,\mathrm{K}_{\mathrm{v}} &= \frac{1}{\mathrm{C}\,\mathrm{N}} \\ n\,\mathrm{M}\,\mathrm{K}_{\mathrm{P}} &= \frac{1}{\mathrm{C}}\left(\frac{1}{\mathrm{N}}+1\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (24.)$$

That is to say, the specific heat of unity of volume at constant volume is inversely proportional to the fraction by which the ratio of the two specific heats exceeds unity; a conclusion already deduced from experiment by DULONG.

The following is a comparison of the ratios of the apparent specific heats under constant pressure, of unity of volume of oxygen and hydrogen respectively, to that of atmospheric air, as deduced from Equation (24.), with those determined experimentally by DE LA ROCHE and BÉRARD.

	Ratio $\frac{n\,\mathrm{M}\,\mathrm{K}_{\mathrm{P}}\ (\mathrm{Gas})}{n\,\mathrm{M}\,\mathrm{K}_{\mathrm{P}}\ (\mathrm{Atmos.\ air})}$	
Gas.	By Theory.	By Experiment.
Oxygen, ...	0·973	0·9765
Hydrogen, ...	0·973	0·9033

This comparison exhibits a much more close agreement between theory and experiment than has been hitherto supposed to exist, the errors in the constants employed having had the effect of making the ratio $1+\mathrm{N}$ seem greater for atmospheric air than for oxygen and hydrogen, while in fact it is smaller.

To treat the other substances on which both M. DULONG and MM. DE LA

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* The following are some additional determinations of the value of γ for atmospheric air, founded upon experiments on the velocity of sound:—

Observers.	T Centigrade.	a Mètres per second.	γ
Bravais and Martins: mean of several experiments at temperatures varying from 5° to 11° centigrade, reduced to 0° (Comptes Rendus, xix.) . .	0°	332·37	1·40955
Moll and Van Beek: reduced to	0°	332·25	1·40853
Stampfer and Myrbach: reduced to 0° (not corrected for moisture)	0°	332·96	1·41456
Académie des Sciences, 1738: (not corrected for moisture)	6°·1	337·10	1·418

A variation of one mètre per second in the velocity of sound at 0° corresponds to a variation of ·0085 in the value of γ.

ROCHE and BÉRARD made experiments as perfect gases, would lead to sensible errors. I have, therefore, confined my calculations for the present to oxygen, hydrogen, and atmospheric air.

(13.) The heat produced by compressing so much of a perfect gas as would occupy unity of volume under the pressure unity, at the temperature 0° centigrade, from its actual volume $n \mathrm{M} \mathrm{V}_1 = \frac{\tau}{\mathrm{P}\,\mathrm{C}}$ into a volume which is less in a given ratio s (when κ is neglected as compared with τ), is expressed by the following motion :—

$$n \mathrm{M} Q = -\frac{1}{\mathrm{C}} \int_{\mathrm{V}_1}^{s \mathrm{V}_1} d\mathrm{V} \,.\, \frac{\tau}{\mathrm{V}} = -n \mathrm{M} \mathrm{V}_1 \int_1^s \mathrm{P}\, ds \quad . \quad . \quad . \quad (25.)$$

being, in fact, equal to the mechanical power used in the compression. When the temperature is maintained constant, this becomes

$$n \mathrm{M} Q_{(\tau)} = \frac{\tau}{\mathrm{C}} \log_e \,.\, \frac{1}{s} \quad . \quad . \quad . \quad . \quad (26.)$$

which is obviously independent of the nature of the gas.

Hence *equal volumes of all substances in the state of perfect gas, at the same pressure and at equal and constant temperatures, being compressed by the same amount, disengage equal quantities of heat;* a law already deduced from experiment by DULONG.

(14.) The determination of the fraction N affords the means of calculating the mechanical or absolute value of specific heat, as defined by Equation 1, Section First. The data for atmospheric air being taken as follows:—

$$\mathrm{N} = 0{\cdot}4, \qquad \mathrm{C} = 274^{\circ}{\cdot}6 \text{ centigrade},$$

$\frac{1}{n \mathrm{M}}$ = height of an imaginary column of air of uniform density, at the temperature 0° cent., whose pressure by weight on a given base is equal to its pressure by elasticity, ... $= 7990$ mètres,
$= 26214$ feet:—

the real specific heat of atmospheric air, or the depth of fall equivalent to one centigrade degree of temperature in that gas, is found to be

$$k = \frac{.1}{\mathrm{C}\, n\, \mathrm{M}\, \mathrm{N}} = 72{\cdot}74 \text{ mètres} = 238{\cdot}66 \text{ feet} \quad . \quad . \quad . \quad (27.)$$

The apparent specific heat of atmospheric air, under constant pressure, according to DE LA ROCHE and BÉRARD, is equal to that of liquid water at 0° centigrade × 0·2669. The ratio of its real specific heat to the apparent specific heat of water at 0° centigrade, is, therefore,

$$\cdot 2669 \times \frac{1{\cdot}0}{1.4} = \cdot 1906,$$

And, consequently, the mechanical value of the apparent specific heat of liquid water, at the temperature of melting ice, is

$$\left.\begin{array}{l}\dfrac{\mathfrak{k}\,(\text{at. air})}{\cdot 1906}=381\cdot 64 \text{ mètres}=1252 \text{ feet per centigrade degree,}\\ \qquad\qquad \text{or } 695\cdot 6 \text{ feet per degree of Fahrenheit's scale,}\end{array}\right\} \quad . \quad . \quad . \quad (28.)$$

This quantity we shall denote by K_w. It is the mechanical equivalent of the ordinary *thermal unit*.

I have already pointed out (in Article 2. of the First Section) the causes which tend to make the apparent value of the mechanical equivalent of heat, in Mr Joule's experiments, greater than the true value. The differences between the result I have just stated, and those at which he has arrived, do not seem greater than those causes are capable of producing, when combined with the uncertainty of experiments, like those of Mr Joule, on extremely small variations of temperature.

(15.) Besides the conditions of constant volume and constant pressure, there is a third condition in which it is of importance to know the apparent specific heat of an elastic fluid, namely, the condition of vapour at saturation, or in contact with its liquid.

The apparent specific heat of a vapour at saturation, is the quantity of heat which unity of weight of that vapour receives or gives out, while its temperature is increased by one degree, its volume being at the same time compressed so as to bring it to the maximum pressure corresponding to the increased temperature.

It has been usually taken for granted, that this quantity is the same with the variation for one degree of temperature, of what is called the *total heat* of evaporation. Such is, indeed, the case according to the theory of Carnôt; but I shall shew that, according to the mechanical theory of heat, these two quantities are not only distinct, but in general of contrary signs.

I shall, for the present, consider such vapours only as may be treated in practice as perfect gases, so as to make the first of the Equations (20.) applicable.

It has been shewn that the logarithm of the maximum elasticity of a vapour in contact with its liquid may be represented by the expression

$$\log P = a - \frac{\beta}{\tau} - \frac{\gamma}{\tau^2}$$

The coefficients a, β, γ, being those adapted for calculating the common logarithm of the pressure, I shall use the accented letters a', β', γ', to denote those suited to calculate the hyperbolic logarithm, being equal respectively to the former coefficients $\times 2\cdot 3025851$.[*]

Then for vapour at saturation,

$$\frac{dP}{P\,d\tau} = \frac{\beta'}{\tau^2} + \frac{2\gamma'}{\tau^3} \quad . \quad . \quad . \quad . \quad (29.)$$

Making this substitution in the general Equation (21.), we find the following value for the apparent specific heat of perfectly gaseous vapour at saturation :—

$$\left.\begin{aligned} \mathrm{K}_s &= \mathfrak{k} + \mathrm{P}\frac{d\mathrm{V}}{d\tau} = \mathfrak{k}\left(1+\mathrm{N} \cdot \frac{\tau}{\mathrm{V}}\frac{d\mathrm{V}}{d\tau}\right) \\ &= \mathfrak{k}\left\{1+\mathrm{N}\left(1-\frac{\tau}{\mathrm{P}}\frac{d\mathrm{P}}{d\tau}\right)\right\} \\ &= \frac{1}{\mathrm{C}\,n\,\mathrm{M}}\left(\frac{1}{\mathrm{N}}+1-\frac{\beta'}{\tau}-\frac{2\gamma}{\tau^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (30.)$$

(16.) For the vapours of which the properties are known, the negative terms of this expression exceed the positive at all ordinary temperatures, so that the kind of apparent specific heat now under consideration is a negative quantity:— that is to say, that if a given weight of vapour at saturation is increased in temperature, and at the same time maintained by compression at the maximum elasticity, the heat generated by the compression is greater than that which is required to produce the elevation of temperature, and a surplus of heat is given out; and on the other hand, if vapour at saturation is allowed to expand, and at the same time maintained at the temperature of saturation, the heat which disappears in producing the expansion is greater than that set free by the fall of temperature, and the deficiency of heat must be supplied from without, *otherwise a portion of the vapour will be liquefied, in order to supply the heat necessary for the expansion of the rest.*

This circumstance is obviously of great importance in meteorology, and in the theory of the steam-engine. There is as yet no experimental proof of it. It is true, that, in the working of non-condensing engines, it has been found that the steam which escapes is always at the temperature of saturation corresponding to its pressure, and carries along with it a portion of water in the liquid state; but it is impossible to distinguish between the water which has been liquefied by the expansion of the steam, and that which has been carried over mechanically from the boiler.

The calculation of the proportion of vapour liquefied by a given expansion, requires the knowledge of the latent heat of evaporation, which forms the subject of the next section.

SECTION III.—OF THE LATENT AND TOTAL HEAT OF EVAPORATION, ESPECIALLY FOR WATER.

(17.) The latent heat of evaporation of a given substance at a given temperature, is the amount of heat which disappears in transforming unity of weight of the substance from the liquid state, to that of vapour of the maximum density for the given temperature, being consumed in producing an increase of volume, and an unknown change of molecular arrangement.

It is obvious, that if the vapour thus produced is reconverted into the liquid state *at the same temperature,* the heat given out during the liquefaction must be equal to that consumed during the evaporation; for as the sum of the expansive and compressive powers, and of those dependent on molecular arrangement during the whole process, is equal to zero, so must the sum of the quantities of heat absorbed and evolved.

The heat of liquefaction, at a given temperature, is therefore equal to that of evaporation, with the sign reversed.

(18.) If to the latent heat of evaporation at a given temperature, is added the quantity of heat necessary to raise unity of weight of the liquid from a certain fixed temperature (usually that of melting ice) to the temperature at which the evaporation takes place, the result is called the total heat of evaporation from the fixed temperature chosen.

According to the theory of CARNÔT, this quantity is called the constituent heat of vapour; and it is conceived, that if liquid at the temperature of melting ice be raised to any temperature and evaporated, and finally brought in the state of vapour to a certain given temperature, the whole heat expended will be equal to the constituent heat corresponding to that given temperature, and will be the same, whatsoever may have been the intermediate changes of volume, or the temperature of actual evaporation.

According to the mechanical theory of heat, on the other hand, the quantity of heat expended must vary with the intermediate circumstances; for otherwise no power could be gained by the alternate evaporation and liquefaction of a fluid at different temperatures.

(19.) The law of the latent and total heat of evaporation is immediately deducible from the principle of the constancy of the total *vis viva* in the two forms of heat and expansive power, when the body has returned to its primitive density and temperature, as already laid down in Article 7.

That principle, when applied to evaporation and liquefaction, may be stated as follows:—

Let a portion of fluid in the liquid state be raised from a certain temperature to a higher temperature: let it be evaporated at the higher temperature: let the vapour then be allowed to expand, being maintained always at the temperature of saturation for its density, until it is restored to the original temperature, at which temperature let it be liquefied:—then *the excess of the heat absorbed by the fluid above the heat given out, will be equal to the expansive power generated.*

To represent those operations algebraically,—let the lower absolute temperature be τ_0:—the volume of unity of weight of liquid at that temperature, v_0, and that of vapour at saturation, V_0: let the pressure of that vapour be P_0: the latent heat of evaporation of unity of weight, L_0; and let the corresponding quantities for the higher absolute temperature τ_1, be v_1, V_1, P_1, L_1. Let K_L represent the mean apparent specific heat of the substance in the liquid form between the temperatures τ_0 and τ_1. Then,—

First, Unity of weight of liquid being raised from the temperature τ_0 to the temperature τ_1, absorbs the heat,

$$K_L\,(\tau_1-\tau_0)$$

and produces the expansive power,

$$\int_{v_0}^{v_1} d\,v\,.\,P$$

Secondly, It is evaporated at the temperature τ_1, absorbing the heat

$$L_1$$

and producing the expansive power,

$$P_1\,(V_1-v_1)$$

Thirdly, The vapour expands, at saturation, until it is restored to the original temperature τ. In this process it absorbs the heat,

$$-\int_{\tau_0}^{\tau_1} d\,\tau\,.\,K_s$$

and produces the expansive power,

$$\int_{V_1}^{V_0} d\,V\,.\,P$$

Fourthly, It is liquefied at the original temperature, giving out the heat

$$L_0$$

and consuming the compressive power,

$$P_0\,(V_0-v_0).$$

The equation between the heat which has disappeared, and the expansive power which has been produced, is as follows:—

$$\left.\begin{aligned} & L_1-L_0+K_L\,(\tau_1-\tau_0)-\int_{\tau_0}^{\tau_1} d\,\tau\,.\,K_s \\ & =P_1\,(V_1-v_1)-P_0\,(V_0-v_0)+\int_{v_0}^{v_1} d\,v\,.\,P+\int_{V_1}^{V_0} d\,V\cdot P \end{aligned}\right\} \quad . \quad . \quad . \quad (31.)$$

If the vapour be such that it can be regarded as a perfect gas without sensible error, the substitution of $\mathfrak{k}+P\frac{d\,V}{d\,\tau}$ for K_s, and of $\frac{\tau}{C\,n\,M}=\mathfrak{k}\,N\,\tau$ for $P\,V$, transforms the above to

$$\left.\begin{aligned} & L_1-L_0+\{K_L-\mathfrak{k}\,(1+N)\}\,(\tau_1-\tau_0) \\ & =-P_1\,v_1+P_0\,v_0+\int_{v_0}^{v_1} d\,v\,.\,P=-\int_{P_0}^{P_1} d\,P\,.\,v \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (32.)$$

In almost all cases which occur in practice, v is so small as compared with V, that $-\int d\,P\,.\,v$ may be considered as sensibly $=0$; and therefore (sensibly)

$$L_1+K_L\,(\tau_1-\tau_0)=L_0+\mathfrak{k}\,(1+N)\,(\tau_1-\tau_0) \quad . \quad . \quad . \quad (33.)$$

Now this quantity, which I shall denote by H, is the total heat required to raise unity of weight of liquid from τ_0 to τ_1 of absolute temperature, and to evaporate it at the latter temperature. Therefore *the total heat of evaporation, where the vapour may be treated as a perfect gas, increases sensibly at an uniform rate with the temperature of evaporation; and the coefficient of its increase with temperature is equal to the apparent specific heat of the vapour at constant pressure*, $\mathfrak{k}\,(1+N)$.

(20.) There have never been any experiments from which the apparent specific heat of steam under constant pressure can be deduced in the manner in which that of permanent gases has been ascertained.

The experiments of M. Regnault, however, prove that the total heat of evaporation of water increases uniformly with the temperature from 0° to 200° centigrade, and thus far fully confirm the results of this theory.

The coefficient of increase is equal to

$$K_w \times 0{\cdot}305$$

Its mechanical value is consequently

116·4 metres=382 feet per centigrade degree, or
212 feet per degree of Fahrenheit. . . . (34.)

Although the principle of the conservation of *vis viva* has thus enabled us to ascertain the *law of increase* of the total heat of evaporation, it does not enable us to calculate *à priori* the constant L_0 of the formula, being the latent heat of evaporation at the fixed temperature from which the total heat is measured; for the changes of molecular arrangement which constitute evaporation are unknown.

When the fixed temperature is that of melting ice, M. Regnault's experiments give 606·5 centigrade degrees, applied to liquid water as the value of this constant; so that

$$\left.\begin{array}{l} H=K_w\,(606^\circ{\cdot}5+{\cdot}305\,T^\circ) \\ \quad\text{For the centigrade scale,} \\ H=K_w\,(1091^\circ{\cdot}7+{\cdot}305\,(T^\circ-32^\circ)) \\ \quad\text{For Fahrenheit's scale.} \end{array}\right\}\quad\ldots\quad(35.)$$

is the complete expression for the heat required to raise unity of weight of water from the temperature of melting ice to T° above the ordinary zero, and to evaporate it at the latter temperature. This formula has been given by M. Regnault as merely empirical; but we have seen that it closely represents the physical law, when quantities depending on the expansion of water are neglected.

It must be remarked, that the unit of heat in M. Regnault's tables is not precisely the specific heat of water at 0° centigrade, but its mean specific heat between the initial and final temperatures of the water in the calorimeter. The utmost error, however, which can arise from this circumstance, is less than $\frac{1}{1200}$ of the total heat of evaporation, so that it may safely be neglected.

The coefficient ·305 $K_w = 382$ feet per centigrade degree is the apparent specific heat of steam at constant pressure; that is to say, for steam,—

$$\left.\begin{array}{l} \mathfrak{k} + \dfrac{1}{C\,n\,M} = 382 \text{ feet per centigrade degree,} \\ \text{but } \dfrac{1}{C\,n\,M} = 153 \text{ feet.} \\ \text{Therefore the real specific heat of steam is} \\ \mathfrak{k} = \dfrac{1}{C n M N} = 229 \text{ feet per centigrade degree,} \\ \qquad = 127{\cdot}4 \text{ feet per deg. of Fahrenheit,} \\ \qquad = K_w \times {\cdot}183 \\ \text{and } N = \dfrac{153}{229} = \dfrac{2}{3}. \end{array}\right\} \quad . \; . \; . \quad (36.)$$

The quantity $-\int_{P_0}^{P_1} d\,P \,.\, v$ has been neglected, as already explained, in these calculations, on account of its smallness. When $\tau_0 = C$, or the fixed point is 0° centigrade, this integral is nearly equal to

$$-v\,P_1 = -\frac{v}{V_1} \cdot \frac{\tau_1}{C\,n\,M} = -\mathfrak{k}\,N \,.\, \frac{v}{V_1}\,\tau_1 \quad . \; . \; . \quad (37.)$$

which, for steam, is equal to

$$-K_w \times {\cdot}122\,\frac{v}{V_1}\,\tau_1.$$

For a pressure of eight atmospheres,

$$\frac{v}{V_1} = \frac{1}{252} \text{ nearly, } \tau_1 = 445^{\circ}{\cdot}5 \ (T = 170^{\circ}{\cdot}9 \text{ cent.})$$

consequently, $-v\,P_1 = -K_w \times 0^{\circ}{\cdot}22$ cent.

a quantity much less than the limit of errors of observation in experiments on latent heat.

This shews that in practice we are justified in overlooking the influence of the volume of the liquid water on the heat of evaporation.

SECTION IV.—OF THE MECHANICAL ACTION OF STEAM, TREATED AS A PERFECT GAS, AND THE POWER OF THE STEAM-ENGINE.

(21.) In the present limited state of our experimental knowledge of the density of steam at pressures differing much from that of the atmosphere, it is desirable to ascertain whether any material error is likely to arise from treating it as a perfect gas. For this purpose the ratio of the volume of steam at 100° centigrade, under the pressure of one atmosphere, to that of the water which produces it at 4°·1 centigrade, as calculated theoretically on the supposition of steam being a perfect gas, is to be compared with the actual ratio.

The weight of one volume of water at 4°·1 centigrade being taken as unity, that of half a volume of oxygen at 0° centigrade, under the pressure of one atmosphere, according to the experiments of M. Regnault, is 0·000714900

That of one volume of hydrogen,	0·000089578
The sum being	0·000804478

The reciprocal of this sum being multiplied by $\frac{374\cdot6}{274\cdot6} = 1\cdot364166$ the ratio of dilatation of a perfect gas from 0° to 100° centigrade) the result gives, for the volume of steam of saturation at 100° centigrade as compared with that of water at

4°·1	1695·72
And for its density,	0·00058972

The agreement of those results with the known volume and density of steam is sufficiently close to shew, that at pressures less than one atmosphere, it may be regarded as a gas sensibly perfect; from which it may be concluded, that in the absence of more precise data, the errors arising from treating it as a perfect gas at such higher pressures as occur in practice, will not be of much importance.

Representing, then, by v the volume of unity of weight of water at 4°·1 centigrade, that of unity of weight of steam at any pressure and temperature will be given by the formula

$$V=\frac{1696\,v\,\varpi}{(\tau)}\cdot\frac{\tau}{P}\quad . \quad . \quad . \quad (38.)$$

ϖ representing the number of units of weight per unit of area in the pressure of one atmosphere, and (τ) the absolute temperature at which the pressure of saturation is one atmosphere; being for the centigrade scale 374°·6, and for Fahrenheit's scale 674°·28.

The mechanical action of unity of weight of steam at the temperature τ and pressure P, during its entrance into a cylinder, before it is permitted to expand, is represented by the product of its pressure and volume, or by

$$P\,V=\frac{1696\,v\,\varpi}{(\tau)}\cdot\tau\quad . \quad . \quad . \quad (39.)$$

The coefficient $\frac{1696\,v\,\varpi}{(\tau)}$ represents a certain depth of fall per degree of absolute temperature, and is the same with the coefficient $\frac{1}{C\,n\,M}$ already referred to. By taking the following values of the factors:—

$v=0\cdot016$ cubic foot per pound avoirdupois,
$\varpi=2117$ pounds avoirdupois per square foot,

we find this coefficient to be

$$\left.\begin{array}{l}153\cdot35 \text{ feet}=46\cdot74 \text{ mètres per centigrade degree,}\\ 85\cdot19 \text{ feet per degree of Fahrenheit;}\end{array}\right\}\quad . \quad . \quad (40.)$$

this determination may be considered correct to about $\frac{1}{1200}$ part. When French measures are used in the calculation, the following is the result :—

$v=1$ cubic centimètre per gramme,

$\varpi=1033{\cdot}3$ grammes per square centimètre,

$$\left.\begin{aligned} \frac{1}{\mathrm{C}\,n\,\mathrm{M}} &= 46{\cdot}78 \text{ mètres per centigrade degree,} \\ &= 153{\cdot}48 \text{ feet } \quad \cdots \quad \cdots \quad \cdots \\ &\quad \text{or } 85{\cdot}27 \text{ feet per degree of Fahrenheit.} \end{aligned}\right\} \quad . \;\; . \;\; . \quad (41.)$$

The difference, which is of no practical importance in calculating the power of the steam-engine, arises in the estimation of the density of liquid water.

(22.) Unit of weight of steam at saturation, of the elasticity P_1 and volume V_1 corresponding to the absolute temperature τ_1, being cut off from external sources of heat, it is now to be investigated what amount of power it will produce in expanding to a lower pressure P_2 and temperature τ_2.

It has already been shewn, at the end of the second section, that if vapour at saturation is allowed to expand, it requires a supply of heat from without to maintain it at the temperature of saturation, otherwise a portion of it must be liquefied to supply the heat required to expand the rest. Hence, when unity of weight of steam at saturation, at the pressure P_1 and volume V_1, expands to a lower pressure P, being cut off from external sources of heat, it will not occupy the entire volume V corresponding to that pressure, according to Equation (38.), but a less volume

$$\mathrm{S}=m\,\mathrm{V},$$

where m represents the weight of water remaining in the gaseous state, the portion $1-m$ having been liquefied during the expansion of the remainder. The expansive action of the steam will therefore be represented by

$$\int_{\mathrm{V}_1}^{\mathrm{S}_2} d\,\mathrm{S}\,.\,\mathrm{P} \quad . \;\; . \;\; . \quad (42.)$$

The law of variation of the fraction m flows from the following considerations:—

Let $\delta\,m$ represent the indefinitely small variation of m corresponding to the indefinitely small change of temperature $\delta\,\tau$; L, the latent heat of evaporation of unity of weight ; K_s, as in Equation (30.), the specific heat of vapour at saturation, which is a negative coefficient varying with the temperature ; then we must have

$$-\mathrm{L}\,\delta\,m=m\,\mathrm{K}_s\,\delta\,\tau, \text{ or } \frac{\delta\,m}{m}=-\frac{\mathrm{K}_s}{\mathrm{L}}\,\delta\,\tau,$$

in order that the heat produced by the liquefaction of $\delta\,m$ may be equal to the heat required to expand m. Hence making, according to Equation (30.)—

$$\mathrm{K}_s\,\delta\,\tau=\mathfrak{k}\,(\delta\,\tau+\mathrm{N}\,\frac{\tau}{\mathrm{V}}\,\delta\,\mathrm{V})$$

and

$$\delta\tau = -\frac{\tau}{\mathrm{V}}\,\delta\mathrm{V}\,\frac{1}{\frac{\beta'}{\tau}+\frac{2\gamma'}{\tau^2}-1}$$

we obtain

$$\frac{\delta m}{m} = -\frac{\mathfrak{k}\,\tau}{\mathrm{L}}\left(\mathrm{N}-\frac{1}{\frac{\beta'}{\tau}+\frac{2\gamma'}{\tau^2}-1}\right)\frac{\delta\mathrm{V}}{\mathrm{V}} \quad . \quad . \quad . \quad (43.)$$

and denoting the coefficient of $\frac{\delta\mathrm{V}}{\mathrm{V}}$ by $-\nu$,

$$\left.\begin{array}{l} \frac{d\log m}{d\log\mathrm{V}} = -\nu\,;\ \frac{d\log\mathrm{S}}{d\log\mathrm{V}} = 1-\nu \\ \text{and because}\quad \frac{d\log\mathrm{V}}{d\log\mathrm{P}} = -1+\frac{1}{\frac{\beta'}{\tau}+\frac{2\gamma}{\tau^2}} \\ \frac{d\log m}{d\log\mathrm{P}} = +\nu\left(1-\frac{1}{\frac{\beta'}{\tau}+\frac{2\gamma}{\tau^2}}\right) \\ \frac{d\log\mathrm{S}}{d\log\mathrm{P}} = -(1-\nu)\left(1-\frac{1}{\frac{\beta'}{\tau}+\frac{2\gamma}{\tau^2}}\right) = -\sigma \end{array}\right\} \quad . \quad . \quad . \quad (44.)$$

As the mean temperature of the liquid thus produced more or less exceeds that of the remaining vapour, a small fraction of it will be reconverted into vapour, if the expansion is carried on slowly enough; but its amount is so small, that to take it into account would needlessly complicate the calculation, without making it to any material extent more accurate.

(23.) The extreme complexity of the exponent σ, considered as a function of the pressure P, would render a general formula for the expansive action $\int \mathrm{P}\,d\,\mathrm{S}$ very cumbrous in its application. For practical purposes, it is sufficient to consider the exponent σ as constant during the expansion which takes place in any given engine, assigning it an average value suitable to the part of the scale of pressures in which the expansion takes place. For engines in which the steam is introduced at pressures not exceeding four atmospheres, I conceive that it will be sufficiently accurate to make

$$\sigma = \frac{6}{7};$$

while for engines in which the initial pressure lies between four and eight atmospheres, the suitable value is

$$\sigma = \frac{5}{6}.$$

The utmost error which can arise from using these exponents is about $\frac{1}{100}$ of the whole power of the engine, and that only in extreme cases.

Making, therefore,

$$\mathrm{P} = \mathrm{P}_1\left(\frac{\mathrm{S}}{\mathrm{V}_1}\right)^{-\frac{1}{\sigma}}$$

we obtain for the value of the expansive action of unity of weight of steam,

$$\left.\begin{aligned} \int_{V_1}^{S_2} d\,S \;.\; P &= P_1\,V_1\,\frac{\sigma}{1-\sigma}\left(1-\left(\frac{S_2}{V_1}\right)^{1-\frac{1}{\sigma}}\right) \\ &= P_1\,V_1\,\frac{\sigma}{1-\sigma}\left(1-s^{1-\frac{1}{\sigma}}\right) \end{aligned}\right\} \quad . \; . \; . \quad (45.)$$

s being used to denote $\frac{S_2}{V_1}$, or the ratio of the volumes occupied by steam at the end and at the beginning of the expansion respectively.

A table to facilitate the computation is given in the Appendix.

The gross mechanical action of unity of weight of steam on one side of the piston is found by adding to the above quantity the action of the steam before it begins to expand, or $P_1\,V_1$, and is therefore

$$P_1\,V_1\left(\frac{1}{1-\sigma}-\frac{\sigma}{1-\sigma}\,s^{1-\frac{1}{\sigma}}\right) \quad . \; . \; . \quad (46.)$$

The values of the coefficients and exponent being

For initial pressures between	$\frac{1}{1-\sigma}$	$\frac{\sigma}{1-\sigma}$	$1-\frac{1}{\sigma}$
1 and 4 atmospheres, ...	7	6	$-\frac{1}{6}$
4 and 8 atmospheres, ...	6	5	$-\frac{1}{5}$

(24.) The following deductions have to be made from the gross action, in order to obtain the action effective in overcoming resistance.

First, For loss of power owing to a portion of the steam being employed in filling steam-passages, and the space called the *clearance* of the cylinder at one end. Let the bulk of steam so employed be the fraction $c\,S_2$ of the space filled by steam at the end of the expansion; then the loss of power from this cause is

$$P_1\,c\,S_2 = c\,s\,P_1\,V_1.$$

Secondly, For the pressure on the opposite side of the piston, of the steam which escapes into the condenser, or into the atmosphere, as the case may be. Let P_3 be the pressure of this steam; the deduction to be made for its action is

$$P_3\,S_2\,(1-c) = P_3\,V_1\,(1-c)\,s.$$

These deductions having been made, there is obtained for the effect of unity of weight of water evaporated,

$$V_1\left\{P_1\left(\frac{1}{1-\sigma}-\frac{\sigma}{1-\sigma}\,s^{1-\frac{1}{\sigma}}-c\,s\right)-P_3\,(1-c)\,s\right\} \quad . \; . \quad (47.)$$

(25.) The effect of the engine in unity of time is found by multiplying the above quantity by the number of units of weight of water evaporated in unity of time.

If this number be denoted by W,

$$\mathrm{W\,S}_2\,(1-c)=\mathrm{W\,V}_1\,(1-c)\,s=\mathrm{A}\,u \quad . \quad . \quad . \quad (48.)$$

will represent the cubical space traversed by the piston in unity of time, A denoting the area of the piston, and u its mean velocity.

Now let the whole resistance to be overcome by the engine be reduced by the principles of statics to a certain equivalent pressure per unit of area of piston, and let this pressure be denoted by R. Then,

$$\mathrm{R\,A}\,u=\mathrm{R\,W\,V}_1\,(1-c)\,s \quad . \quad . \quad . \quad (49.)$$

expresses the effect of the engine in terms of the gross resistance.

We have now the means of calculating the circumstances attending the working of a steam-engine according to the principle of the conservation of *vis viva*, or, in other words, of the equality of power and effect, which regulates the action of all machines that move with an uniform or periodical velocity.

This principle was first applied to the steam-engine by the COUNT DE PAMBOUR; and accordingly, the formulæ which I am about to give only differ from those of his work in the expressions for the maximum pressure at a given temperature, and for the expansive action of the steam, which are results peculiar to the theory of this essay.

In the first place, the effect, as expressed in terms of the pressure, is to be equated to the effect as expressed in terms of the resistance, as follows:—

$$\mathrm{R\,A}\,u=\mathrm{R\,W\,V}_1\,(1-c)\,s=\mathrm{W\,V}_1\left\{\mathrm{P}_1\left(\frac{1}{1-\sigma}-\frac{\sigma}{1-\sigma}\,s^{1-\frac{1}{\sigma}}-c\,s\right)-\mathrm{P}_3\,(1-c)\,s\right\} \quad . \quad . \quad (50.)$$

This is the fundamental equation of the action of the steam-engine, and corresponds with Equation A. of M. DE PAMBOUR's theory.

(26.) Dividing both sides of Equation (50.) by the space traversed by the piston in unity of time $\mathrm{W\,V}_1\,(1-c)\,s$, and transferring the pressure of the waste steam, P_3, to the first side, we obtain this equation:—

$$\mathrm{R}+\mathrm{P}_3=\mathrm{P}_1\,\frac{\frac{1}{1-\sigma}-\frac{\sigma}{1-\sigma}\,s^{1-\frac{1}{\sigma}}-c\,s}{(1-c)\,s} \quad . \quad . \quad . \quad (51.)$$

which gives the means of determining the pressure P_1 at which the steam must enter the cylinder, in order to overcome a given resistance and counter-pressure with a given expansion; or supposing the expansion s to be variable at pleasure, and the initial pressure P_1 fixed, the equation gives the means of finding, by approximation, the expansion best adapted to overcome a given resistance and counter-pressure.

The next step is to determine, from Equations XV. of the Introduction and

(38.) of this section, the volume V_1 of unity of weight of steam corresponding to the maximum pressure P_1. Then Equation (48.) gives the space traversed by the piston in unity of time, which, being multiplied by the resistance R per unit of area of piston, gives the gross effect of the engine.

(27.) If, on the other hand, the space traversed by the piston in unity of time is fixed, Equation (48.) gives the means of determining, from the evaporating power of the boiler W, either the volume V_1 of unity of weight of steam required to work the engine at the given velocity with a given expansion, or the expansion s proper to enable steam of a given initial density to work the engine at the given velocity. The initial pressure P_1 being then determined from the volume V_1, the resistance which the engine is capable of overcoming with the given velocity is to be calculated by means of Equation (51.)

(28.) This calculation involves the determination of the pressure P_1 from the volume V_1 of unity of weight of steam at saturation, which can only be done by approximation. The following formula will be found useful for this purpose:—

$$P_1 = \varpi \left(\frac{V_0}{V_1}\right)^{\frac{12}{11}} \quad . \quad . \quad . \quad (52.)$$

where ϖ represents the pressure of one atmosphere, V_0 the volume of steam of saturation at that pressure (being 1696 times the volume of water at 4°·1 cent., or 27·136 cubic feet per pound avoirdupois), and V_1 the volume of steam of saturation at the pressure P_1. This formula is only applicable between the pressures of one and eight atmospheres: that is to say, when the volume of steam is not greater than 27 cubic feet per pound, nor less than 4, and the temperature not lower than 100° centigrade, nor higher than 171° centigrade (which correspond to 212° and 340° Fahrenheit).

The greatest error in computing the pressure by means of this formula is about $\frac{1}{50}$ of an atmosphere, and occurs at the pressure of four atmospheres, so that it is $\frac{1}{200}$ of the whole pressure. This is sufficiently accurate for practice, in calculating the power of steam-engines; but should a more accurate result be required, the approximate value of the pressure may be used to calculate the temperature by means of Equation XV.; and the temperature thus determined (which will be correct to $\frac{1}{8}$ of a centigrade degree) may then be used in conjunction with the volume to compute a corrected value of the pressure, according to Equation (38.) The pressure, as thus ascertained, will be correct to $\frac{1}{2000}$ of its amount, which may be considered the greatest degree of accuracy attainable.

The most convenient and expeditious mode, however, of computing the pressure from the volume, or *vice versâ*, is by interpolation from the table given in the Appendix to this paper.

(29.) The resistance denoted by R may be divided into two parts; that which arises from the *useful work performed*, and that which is independent of it, being, in fact, the resistance of the engine when unloaded. Now it is evident, that the maximum *useful* effect of the steam has been attained, as soon as it has expanded to a pressure which is in equilibrio with the pressure of the waste steam added to the resistance of the engine when unloaded; for any further expansion, though increasing the total effect, diminishes the useful effect. Therefore if we make

$$R=R'+f,$$

R′ being the resistance arising from the useful work, and f the resistance of the engine when unloaded, both expressed in the form of pressure on the piston, the expansion corresponding to the maximum of useful effect will take place when

$$P_2=P_3+f \quad . \quad . \quad . \quad .$$

the corresponding ratio of expansion being

$$s=\left(\frac{P_1}{P_3+f}\right)^e \quad . \quad . \quad . \quad . \qquad \Bigg\} \quad . \quad . \quad (53.)$$

The maximum useful effect with a given pressure on the safety-valve has been so fully discussed by M. de Pambour, that it is unnecessary to do more than to state that it takes place when the initial pressure in the cylinder is equal to that at the safety-valve: that is to say, when it and the useful resistance are the greatest that the safety-valve will permit.

(30.) Annexed is a table of the values of some of the quantities which enter into the preceding equations in the notation of the Count de Pambour's works.

Expression in the Notation of this paper.	Equivalent Expression in M. de Pambour's Notation.
$R=R'+f$	$(1+\delta)\,r+f$
$A\,u$	$a\,v$
W	S × weight of one cubic foot of water.
P_3	p
s	$\frac{l+c}{l'+c}$
c	$\frac{c}{l'+c}$

(31.) As an illustration, I shall calculate the maximum useful effect of one pound, and of one cubic foot of water, in a Cornish double-acting engine, in the circumstances taken by M. de Pambour as an example for that kind of engine: that is to say,—

Clearance one-twentieth of the stroke, or $c=\frac{1}{21}$

Resistance not depending on the useful load,	$f=$	72 lb. per square foot.
Pressure of condensation, . . .	$P_3=$	576 lb. ... ...

Consequently to give the maximum useful effect,

$P_2 = P_3 + f \qquad = 648$ lb. per square foot.

Total pressure of the steam when first admitted, $P_1 = 7200$ lb. ... ...

Volume of 1 lb. of steam $V_1 = 8{\cdot}7825$ cubic feet.

Therefore $P_1 V_1 = 63234$ lbs. raised one foot.

$$\frac{P_1}{P_2} = \frac{7200}{648} \text{ and consequently,}$$

Expansion to produce the maximum useful effect $s = \left(\frac{P_1}{P_2}\right)^{\frac{6}{7}} = 7{\cdot}877$

Space traversed by the piston during the action of one pound of steam,

$$= V_1 (1-c) s = 65{\cdot}886 \text{ cubic feet.}$$

Gross effect of one pound of steam, in pounds raised one foot high,

$$= P_1 V_1 \left(7 - 6s^{-\frac{1}{6}} - \frac{s}{21}\right) - P_3 V_1 (1-c) s \quad . \quad . \quad . \quad = 112004$$

Deduct for resistance of engine when unloaded $f V_1 (1-c) s$. $= 4744$

Effect of one pound of steam in overcoming resistance depending on useful load, ... 107260

This being multiplied by 62½, gives for the effect of one cubic foot of water evaporated, in pounds raised one foot, ... 6,703,750

It is here necessary to observe, that M. DE PAMBOUR distinguishes the useful resistance into two parts, the resistance of the useful load independently of the engine, and the increase in the resistance of the engine, arising from the former resistance, and found by multiplying it by a constant fraction which he calls δ. In calculating the net useful effect, he takes into account the former portion of the resistance only; consequently,

$$\text{Net useful effect as defined by M. DE PAMBOUR} = \frac{\text{Gross effect} - f V_1 (1-c) s}{1+\delta} \quad . \quad (54.)$$

The value of δ, for double acting steam-engines generally, is considered by M. DE PAMBOUR to be $\frac{1}{7}$; consequently, to reduce the effect of one cubic foot of water as calculated above to that which corresponds with his definition, we must deduct $\frac{1}{8}$, which leaves,

5,865,781 lb. raised one foot.

M. DE PAMBOUR's own calculation gives,

6,277,560

being too large by about one-fifteenth.

(32.) In order to shew the limit of the effect which may be expected from the expenditure of a given quantity of heat in evaporating water, and also to verify the approximate method employed in calculating the expansive action of the steam, I shall now investigate the *maximum gross effect*, including resistance of all kinds, producible by evaporating unity of weight of water at a higher temperature and liquefying it at a lower, and compare, in two examples, the power produced, with the heat which disappears during the action of the steam, as calculated directly.

To obtain the maximum gross effect, the steam must continue to act expansively until it reaches the pressure of condensation, so that $P_2 = P_3$. The clearance must also be null, or $c=0$. Making those substitutions in the formula (47.), we find, for the maximum gross effect of unity of weight of water, evaporated under the pressure P_1 and liquefied under the pressure P_2,

$$P_1 V_1 \frac{1}{1-\sigma}\left(1 - s^{1-\frac{1}{\sigma}}\right) = P_1 V_1 \frac{1-\left(\frac{P_2}{P_1}\right)^{1-\sigma}}{1-\sigma} \quad . \quad . \quad . \quad (55.)$$

In order to calculate directly the heat which is converted into power in this operation, let τ_1, τ_2, respectively represent the absolute temperatures of evaporation and liquefaction, and L_2 the latent heat of evaporation at the lower temperature τ_2; then the total heat of evaporation at τ_1, starting from τ_2 as the fixed point, by Equation (33.), is

$$H_{2,1} = L_2 + \cdot 305\, K_w (\tau_1 - \tau_2).$$

This is the heat communicated to the water in raising it from τ_2 to τ_1 and evaporating it. Now a weight $1-m$ of the steam is liquefied during the expansion at temperatures varying from τ_1 to τ_2, so that it may be looked upon as forming a mass of liquid water approximately at the mean temperature $\frac{\tau_1+\tau_2}{2}$, and from which a quantity of heat, approximately represented by

$$K_w (1-m) \frac{\tau_1-\tau_2}{2}$$

must be abstracted, to reduce it to the primitive temperature τ_2.

Finally, the weight of steam remaining, m, has to be liquefied at the temperature τ_2, by the abstraction of the heat

$$m L_2.$$

The difference between the heat given to the water, and the heat abstracted from it, or

$$\left.\begin{aligned} & H_{2,1} - K_w (1-m) \frac{\tau_1-\tau_2}{2} - m L_2 \\ & = (1-m) L_2 + K_w \left(\cdot 305 - \frac{1-m}{2}\right)(\tau_1 - \tau_2) \end{aligned}\right\} \quad . \quad . \quad . \quad (56.)$$

is the heat which has disappeared, and ought to agree with the expression (55.) for the power produced, if the calculation has been conducted correctly.

As a first example, I shall suppose unity of weight of water to be evaporated under the pressure of four atmospheres, and liquefied under that of half an atmosphere; so that the proper values of the coefficients and exponent are

$$\frac{1}{1-\sigma} = 7 \qquad 1-\sigma = \frac{1}{7}$$

The data in this case for calculating the power, are,

$P_1 = 8468$ lb. per square foot.

$V_1 = 7 \cdot 584$ cubic feet for one lb. of steam.

$P_1 V_1 = 64221$ lb. raised one foot.

$\frac{P_2}{P_1} = \frac{1}{8}$, whence $s = 8^{\frac{6}{7}} = 5 \cdot 944$.

Maximum possible effect of one pound of water,

$$= P_1 V_1 \times 7 \left(1 - \left(\frac{1}{8}\right)^{\frac{1}{7}}\right) = 115600 \text{ lb. raised one foot.}$$

Being the mechanical equivalent of 92°·3 centigrade degrees applied to one pound of liquid water at 0° C. ; or,

$$92^\circ \cdot 3\ K_w$$

Maximum possible effect of one cubic foot of water, 7,225,000 lb. raised one foot.

In order to calculate directly the heat converted into power, we have,

$\tau_1 = C + 144^\circ \cdot 1$ cent. $\tau_2 = C + 81^\circ \cdot 7$

$L_2 = 549^\circ \cdot 7\ K_w$

$H_{2,1} = 568^\circ \cdot 7\ K_w$ = heat expended in the boiler.

$1 - m = \cdot 14$ nearly = proportion of steam liquefied during the expansion.

The heat converted into mechanical power, as calculated from these data, is found to be,

$$91^\circ \cdot 6\ K_w$$

differing by only 0°·7 from the amount as calculated from the power produced.

The direct method, however, is much less precise than the other, and is to be regarded as only a verification of the general principle of calculation.

The heat rendered effective, in the above example, is $\frac{92 \cdot 3}{568 \cdot 7}$, or less than *one-sixth* of that expended in the boiler.

As a second example, I shall suppose the steam to be produced at a pressure of eight atmospheres, and to expand to that of one atmosphere. In this case,

$P_1 = 16936$ lbs. per square foot.

$V_1 = 4 \cdot 03$ cubic feet per lb. of steam.

$P_1 V_1 = 68252$ lbs. raised one foot.

$\frac{P_2}{P_1} = \frac{1}{8} \therefore s = 5 \cdot 657 = 8^{\frac{5}{6}}$

Maximum possible effect of one pound of water,

$$= P_1 V_1 \times 6 \left(1 - \left(\frac{1}{8}\right)^{\frac{1}{6}}\right) = 119{,}942 \text{ lb. raised one foot.}$$

Being the equivalent of 95°·8 K_w (Centigrade).

Maximum possible effect of one cubic foot of water = 7,496,375 lb. raised one foot.

The data for calculating directly the heat rendered effective are,

$\tau_1 = C + 170^\circ \cdot 9$ cent. $\tau_2 = C + 100^\circ$

$L_2 = 537^\circ\ K_w$

$H_{2,1} = 558^\circ \cdot 6\ K_w$ = heat expended in the boiler.

$1 - m = \cdot 148$ nearly = steam liquefied during the expansion.

Whence, the heat converted in power, as calculated directly, is,

$$95^{\circ}{\cdot}8\ K_w$$

agreeing with the calculation from the power produced.

In this example, the heat rendered effective is $\frac{95{\cdot}8}{558{\cdot}6}$, or somewhat more than one-sixth of that expended in the boiler.

(33.) The results of the calculations of maximum possible effect, of which examples have just been given, are *limits* which may be approached in practice by Cornish and similar engines, but which cannot be fully realised; and yet it has been shewn, that in those theoretical cases only about *one-sixth* of the heat expended in the boiler is rendered effective. In practice, of course, the proportion of heat rendered effective must be still smaller; and, in fact, in some unexpansive engines, it amounts to only one-twenty-fourth, or even less.

Dr LYON PLAYFAIR, in a memoir on the Evaporating Power of Fuel, has taken notice of the great disproportion between the heat expended in the steam-engine and the work performed. It has now been shewn that this waste of heat is, to a great extent, a necessary consequence of the nature of the machine. It can only be reduced by increasing the initial pressure of the steam, and the extent of the expansive action; and to both of those resources there are practical limits, which have already in some instances been nearly attained.

APPENDIX TO THE FOURTH SECTION,

CONTAINING TABLES TO BE USED IN CALCULATING THE PRESSURE, VOLUME, AND MECHANICAL ACTION OF STEAM, TREATED AS A PERFECT GAS.

The object of the first of the annexed tables is to facilitate the calculation of the volume of steam of saturation at a given pressure, of the pressure of steam of saturation at a given volume, and of its mechanical action at full pressure.

The pressures are expressed in pounds avoirdupois per square foot, and the volumes by the number of cubic feet occupied by one pound avoirdupois of steam, when considered as a perfect gas; those denominations being the most convenient for mechanical calculations in this country.

The columns to be used in determining the pressure from the volume, and *vice versâ*, are the third, fourth, sixth, and seventh.

The third column contains the common logarithms of the pressures of steam of saturation for every fifth degree of the centigrade thermometer from -30° to $+260^{\circ}$: that is to say, for every ninth degree of Fahrenheit's thermometer from -22° to $+500^{\circ}$.

The fourth column gives the differences of the successive terms of the third column.

The sixth column contains the common logarithms of the volume of one pound of steam of saturation corresponding to the same temperatures.

The seventh column contains the differences of the successive terms of the sixth column, which are negative; for the volumes diminish as the pressures increase.

By the ordinary method of taking proportional parts of the differences, the logarithms of the volumes corresponding to intermediate pressures, or the logarithms of the pressures corresponding to intermediate volumes, can be calculated with great precision. Thus, let $X+h$ be the logarithm of a pressure not found in the table, X being the next less logarithm which is found in the table; let Y be the logarithm of the volume corresponding to X, and $Y-k$ the logarithm of the volume corresponding to $X+h$; let H be the difference between X and the next greater logarithm in the table, as given in the fourth column, and K the corresponding difference in the seventh column; then by the proportion

$$H : K :: h : k$$

either $Y-k$ may be found from $X+h$, or $X+h$ from $Y-k$.

In the fifth and eighth columns respectively, are given the actual pressures and volumes corresponding to the logarithms in the third and sixth columns, to five places of figures.

In the ninth column are given the values of the quantity denoted by $P_1 V_1$ in the formulæ, which represents the mechanical action of unity of weight of steam at full pressure, or before it has begun to expand, in raising an equal weight. Those values are expressed in feet, being the products of the pressures in the fifth column by the volumes in the eighth, and have been found by multiplying the absolute temperature in centigrade degrees by 153·48 feet. Intermediate terms in this column, for a given pressure or a given volume, may be approximated to by the method of differences, the constant difference for 5° centigrade being 767·4 feet; but it is more accurate to calculate them by taking the product of the pressure and volume.

When the pressure is given in other denominations, the following logarithms are to be added to its logarithm, in order to reduce it to pounds avoirdupois per square foot:—

For Millimètres of mercury,	0·44477
Inches of mercury,	1·84960
Atmospheres of 760 millimètres,	3·32559
Atmospheres of 30 inches,	3·32672
Kilogrammes on the square centimètre,	3·31136
Kilogrammes on the circular centimètre,	3·41627
Kilogrammes on the square mètre,	$\bar{1}$·31136
Pounds avoirdupois on the square inch,	2·15836
Pounds avoirdupois on the circular inch,	2·26327

To reduce the logarithm of the number of cubic mètres occupied by one kilogramme to that of the number of cubic feet occupied by one pound avoirdupois, add 1·20463.

The logarithms are given to five places of decimals only, as a greater degree of precision is not attainable in calculations of this kind.

The second table is for the purpose of calculating the mechanical action of steam in expansive engines.

The first column contains values of the fraction of the entire capacity of the cylinder which is filled with steam before the expansion commences (being the quantity $\frac{1}{s}$ of the formulæ), for every hundredth part, from 1·00, or the whole cylinder, down to 0·10, or one-tenth.

If l be the entire length of stroke, l' the portion performed at full pressure, and c the fraction of the entire capacity of the cylinder allowed for clearance, then

$$\frac{l'}{l} = \frac{\frac{1}{s} - c}{1 - c}, \quad \text{and} \quad \frac{1}{s} = (1 - c)\frac{l'}{l} + c.$$

The *entire capacity of the cylinder* is to be understood to include clearance at one end only.

The second column gives the reciprocals of the quantities in the first, or the values of the ratio of expansion s.

The third and fourth columns, headed Z, give the values of the quantity $\frac{1}{1-\sigma} - \frac{\sigma}{1-\sigma} s^{1-\frac{1}{\sigma}}$ of Article 23, which represents the ratio of the entire gross action of the steam to its action at full pressure, without allowing for clearance. The third column is to be used for initial pressures of from one to four atmospheres; and the fourth for initial pressures of from four to eight atmospheres.

The deduction to be made from the quantity Z for clearance is $c\,s$, or the product of the fraction of the cylinder allowed for clearance by the ratio of expansion. Hence, to calculate from the tables the net mechanical action of unity of weight of steam, allowing for the counter-pressure of the waste steam P_3, as well as for clearance, we have the formula

$$P_1 V_1 (Z - c\,s) - P_3 V_1 (1 - c)\,s$$

being equivalent to the formula (47.) of this paper.

TABLE I.—*Pressure and Volume of Steam, and its Action at Full Pressure.*

(1.) Temperature Fahrenheit.	(2.) Temperature Centigrade.	(3.) Logarithm of pressure in lbs. per square foot.	(4.) Differences.	(5.) Pressure in lbs. per square foot.	(6.) Logarithm of volume of one lb. of Steam in cubic feet.	(7.) Differences.	(8.) Volume of one lb. of Steam in cubic feet.	(9.) Action of a given weight of Steam in raising an equal weight in feet, at full pressure $=P_1 V_1$
−22°	−30°	$\bar{1}$·99278	0·20563	0·9835	4·58173	0·19684	38171	37541
−13	−25	0·19841	0·19602	1·5791	4·38489	0·18741	24260	38309
− 4	−20	0·39443	0·18710	2·4799	4·19748	0·17865	15757	39076
+ 5	−15	0·58153	0·17864	3·8153	4·01883	0·17036	10443	39843
14	−10	0·76017	0·17085	5·7567	3·84847	0·16272	7054·6	40611
23	− 5	0·93102	0·16348	8·5314	3·68575	0·15550	4850·1	41378
32	0	1·09450	0·15661	12·431	3·53025	0·14877	3390·4	42146
41	+ 5	1·25111	0·15012	17·828	3·38148	0·14242	2407·0	42913
50	10	1·40123	0·14404	25·190	3·23906	0·13648	1734·0	43680
59	15	1·54527	0·13836	35·097	3·10258	0·13093	1266·4	44448
68	20	1·68363	0·13284	48·265	2·97165	0·12553	936·81	45215
77	25	1·81647	0·12780	65·535	2·84612	0·12061	701·65	45983
86	30	1·94427	0·12297	87·957	2·72551	0·11590	531·51	46750
95	35	2·06724	0·11842	116·75	2·60961	0·11146	407·01	47517
104	40	2·18566	0·11410	153·34	2·49815	0·10725	314·88	48285
113	45	2·29976	0·11002	199·42	2·39090	0·10328	245·98	49052
122	50	2·40978	0·10614	256·91	2·28762	0·09950	193·92	49820
131	55	2·51592	0·10247	328·04	2·18812	0·09593	154·21	50587
140	60	2·61839	0·09897	415·33	2·09219	0·09253	123·65	51354
149	65	2·71736	0·09566	521·63	1·99966	0·08931	99·922	52122
158	70	2·81302	0·09250	650·16	1·91035	0·08625	81·349	52889
167	75	2·90552	0·08953	804·49	1·82410	0·08336	66·696	53657
176	80	2·99505	0·08658	988·67	1·74074	0·08050	55·048	54424
185	85	3·08163	0·08388	1206·8	1·66024	0·07788	45·734	55191
194	90	3·16551	0·08129	1463·9	1·58236	0·07538	38·226	55959
203	95	3·24680	0·07879	1765·2	1·50698	0·07295	32·135	56726
212	100	3·32559	0·07640	2116·4	1·43403	0·07064	27·166	57494
221	105	3·40199	0·07415	2523·4	1·36339	0·06847	23·088	58261
230	110	3·47614	0·07196	2993·2	1·29492	0·06635	19·721	59028
239	115	3·54810	0·06988	3532·6	1·22857	0·06434	16·927	59796
248	120	3·61798	0·06788	4149·3	1·16423	0·06241	14·596	60563
257	125	3·68586	0·06597	4851·3	1·10182	0·06057	12·642	61331
266	130	3·75183	0·06414	5647·2	1·04125	0·05881	10·996	62098
275	135	3·81597	0·06238	6545·9	0·98244	0·05711	9·6037	62865
284	140	3·87835	0·06069	7557·0	0·92533	0·05548	8·4204	63633
293	145	3·93904	0·05907	8690·4	0·86985	0·05393	7·4105	64400
302	150	3·99811	0·05751	9956·6	0·81592	0·05242	6·5452	65168
311	155	4·05562	0·05601	11366	0·76350	0·05099	5·8010	65935
320	160	4·11163	0·05456	12931	0·71251	0·04959	5·1583	66702
329	165	4·16619	0·05319	14662	0·66292	0·04828	4·6017	67470
338	170	4·21938	0·05184	16572	0·61464	0·04698	4·1176	68237
347	175	4·27122	0·05056	18673	0·56766	0·04576	3·6954	69005
356	180	4·32178	0·04932	20979	0·52190	0·04457	3·3258	69772
365	185	4·37110	0·04812	23502	0·47733	0·04342	3·0014	70539
374	190	4·41922	0·04696	26256	0·43391	0·04231	2·7159	71307
383	195	4·46618	0·04586	29254	0·39160	0·04126	2·4638	72074
392	200	4·51204	0·04478	32512	0·35034	0·04023	2·2405	72842
401	205	4·55682	0·04375	36043	0·31011	0·03924	2·0423	73609
410	210	4·60057	0·04274	39863	0·27087	0·03829	1·8663	74376
419	215	4·64331	0·04176	43986	0·23258	0·03734	1·7084	75144
428	220	4·68507	0·04085	48425	0·19524	0·03649	1·5676	75911
437	225	4·72592	0·03994	53201	0·15875	0·03561	1·4413	76679
446	230	4·76586	0·03906	58326	0·12314	0·03478	1·3278	77446
455	235	4·80492	0·03819	63815	0·08836	0·03395	1·2256	78213
464	240	4·84311	0·03740	69680	0·05441	0·03320	1·1335	78981
473	245	4·88051	0·03660	75947	0·02121	0·03244	1·0501	79748
482	250	4·91711	0·03582	82625	$\bar{1}$·98877	0·03170	0·97447	80516
491	255	4·95293	0·03507	89728	$\bar{1}$·95707	0·03099	0·90588	81283
500	260	4·98800		97275	$\bar{1}$·92608		0·84349	82050

TABLE II.—*Expansive Action of Steam.*

(1.)	(2.)	(3.)	(4.)	(1.)	(2.)	(3.)	(4.)
Fraction of Cylinder filled with Steam at full pressure $=\frac{1}{s}$	Ratio of Expansion $=s$.	Coefficient of gross action $=$ z.		Fraction of Cylinder filled with Steam at full pressure. $=\frac{1}{s}$	Ratio of Expansion. $=s$.	Coefficient of gross action $=$ z.	
		Initial Pressure one to four Atmospheres.	Initial Pressure four to eight Atmospheres.			Initial Pressure one to four Atmospheres.	Initial Pressure four to eight Atmospheres.
1·00	1·000	1·000	1·000	·54	1·852	1·586	1·580
·99	1·010	1·010	1·010	·53	1·887	1·602	1·596
·98	1·020	1·020	1·020	·52	1·923	1·620	1·613
·97	1·031	1·030	1·030	·51	1·961	1·637	1·630
·96	1·042	1·041	1·041	·50	2·000	1·655	1·647
·95	1·053	1·051	1·051	·49	2·041	1·673	1·665
·94	1·064	1·062	1·062	·48	2·083	1·691	1·683
·93	1·075	1·072	1·072	·47	2·128	1·709	1·701
·92	1·087	1·083	1·083	·46	2·174	1·728	1·719
·91	1·099	1·094	1·093	·45	2·222	1·748	1·738
·90	1·111	1·104	1·104	·44	2·273	1·767	1·757
·89	1·124	1·115	1·115	·43	2·326	1·787	1·777
·88	1·136	1·126	1·126	·42	2·381	1·808	1·796
·87	1·149	1·138	1·137	·41	2·439	1·829	1·817
·86	1·163	1·149	1·149	·40	2·500	1·850	1·837
·85	1·176	1·160	1·160	·39	2·564	1·871	1·858
·84	1·190	1·172	1·171	·38	2·632	1·894	1·880
·83	1·205	1·183	1·183	·37	2·703	1·916	1·902
·82	1·220	1·195	1·195	·36	2·778	1·939	1·924
·81	1·235	1·207	1·206	·35	2·857	1·963	1·947
·80	1·250	1·219	1·218	·34	2·941	1·987	1·970
·79	1·266	1·231	1·230	·33	3·030	2·012	1·994
·78	1·282	1·243	1·242	·32	3·125	2·038	2·019
·77	1·299	1·256	1·255	·31	3·225	2·064	2·044
·76	1·316	1·268	1·267	·30	3·333	2·091	2·070
·75	1·333	1·281	1·280	·29	3·448	2·119	2·097
·74	1·351	1·294	1·292	·28	3·571	2·147	2·124
·73	1·370	1·307	1·305	·27	3·704	2·176	2·152
·72	1·389	1·320	1·318	·26	3·846	2·207	2·181
·71	1·408	1·333	1·331	·25	4·000	2·238	2·211
·70	1·429	1·346	1·344	·24	4·167	2·270	2·242
·69	1·449	1·360	1·358	·23	4·348	2·304	2·273
·68	1·471	1·374	1·371	·22	4·545	2·338	2·306
·67	1·493	1·387	1·385	·21	4·762	2·374	2·341
·66	1·515	1·401	1·399	·20	5·000	2·412	2·376
·65	1·538	1·416	1·413	·19	5·263	2·451	2·413
·64	1·563	1·430	1·427	·18	5·556	2·492	2·452
·63	1·587	1·445	1·441	·17	5·882	2·534	2·492
·62	1·613	1·459	1·456	·16	6·250	2·579	2·434
·61	1·640	1·474	1·471	·15	6·667	2·626	2·579
·60	1·667	1·490	1·486	·14	7·143	2·676	2·626
·59	1·695	1·505	1·501	·13	7·692	2·730	2·675
·58	1·724	1·521	1·516	·12	8·333	2·786	2·728
·57	1·754	1·537	1·532	·11	9·091	2·847	2·784
·56	1·786	1·553	1·547	·10	10·000	2·912	2·845
·55	1·818	1·569	1·563				

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